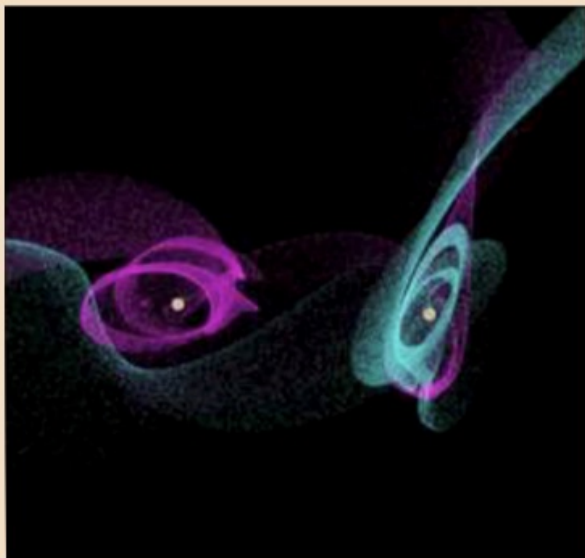
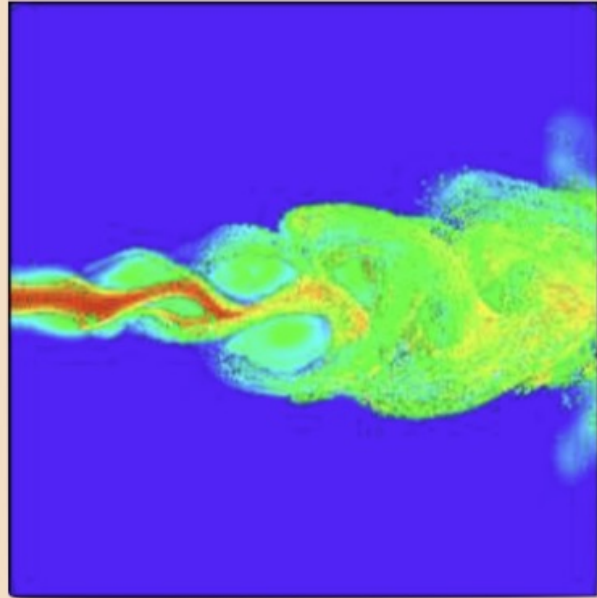
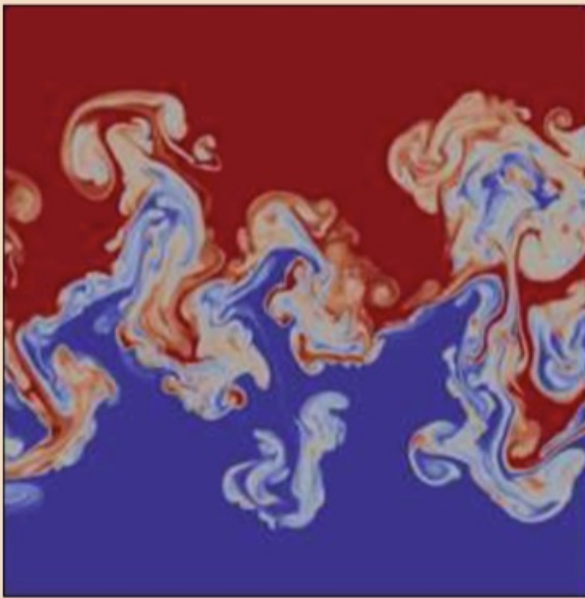


Physical Processes in Astronomy

a concise treatment of fluid dynamics,
collisionless dynamics, plasma physics,
and radiative processes in astrophysics



Frank C. van den Bosch

These lecture notes are constantly being updated, extended and improved

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Course Outline

Astronomy is the study of everything in our Universe outside of the Earth's atmosphere, including the Universe as a whole ('cosmology'). As such, it involves all of physics: quantum mechanics, nuclear and particle physics, classical mechanics, special and general relativity, electromagnetism, statistical physics, hydrodynamics, plasma physics, and even solid state physics. As we will see, though, with the exception of rocky planets and asteroids, all objects in the Universe can be characterized as some kind of **fluid**. In addition, almost all information we receive from these objects reaches us in the form of **radiation** (photons)¹. Consequently, in this course on physical processes in astronomy we will focus almost exclusively on **fluid dynamics** and **radiative processes**. Note that we focus exclusively on neutral fluids; the physics of electrically charged fluids, known as plasmas will not be covered.

We start in Part I with standard hydrodynamics, which applies mainly to neutral, collisional fluids. We start by discussing the continuity, momentum and energy equations, for both ideal and non-ideal fluids. Next we discuss a variety of different flows; vorticity, incompressible barotropic flow, viscous flow, accretion flow, and turbulent flow, before addressing fluid instabilities and shocks. In Part II, we briefly focus on collisionless dynamics. We start by discussing potential theory and the Virial theorem, followed by the Collisionless Boltzmann Equation, from which we derive the Jeans equations. We then study how these Jeans equations differ from the Euler equations that describe ideal, collisional fluids, and briefly discuss orbit theory, integrals of motion, and the Jeans theorem. We end with a treatment of gravitational interactions among collisionless systems. Finally, in Part III we discuss radiative processes and the interaction of light with matter (scattering & absorption), including, among others, Compton scattering, recombination, photo- and collisional ionization, free-free emission, synchrotron emission, radiative transfer and the Saha equation.

It is assumed that the student is familiar with vector calculus, with curvi-linear coordinate systems. A brief overview of these topics is provided in Appendices A-E. The other appendices present detailed background information that is provided for the interested student, but which is not considered part of the course material.

¹Other astrophysical messengers include neutrinos, cosmic rays and gravitational waves

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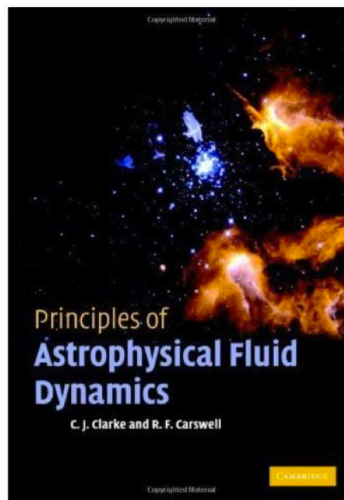
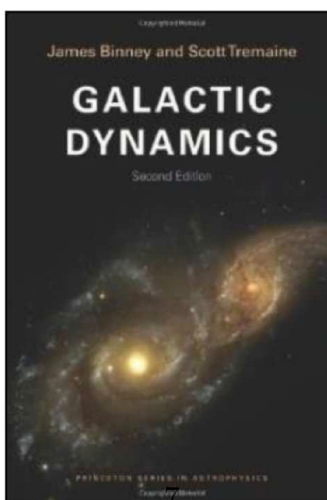
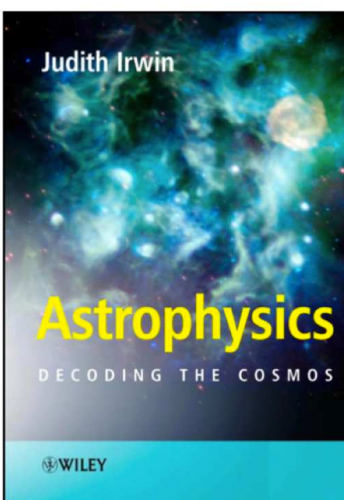
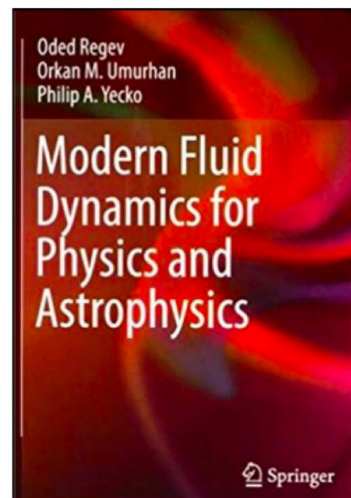
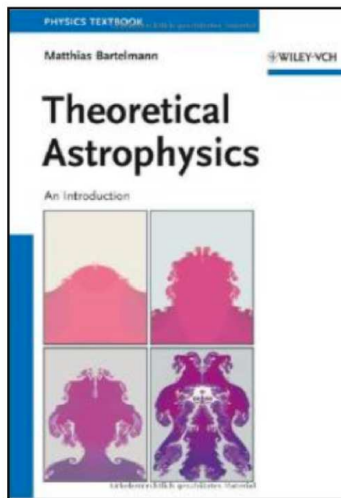
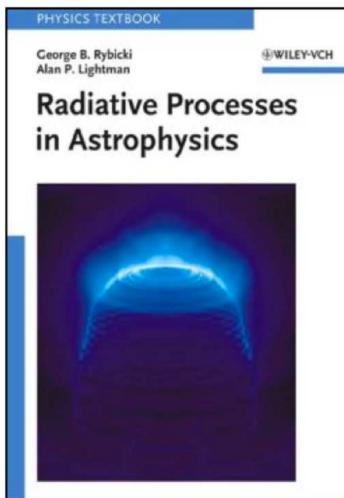
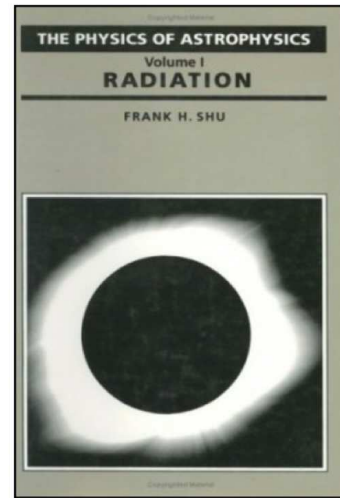
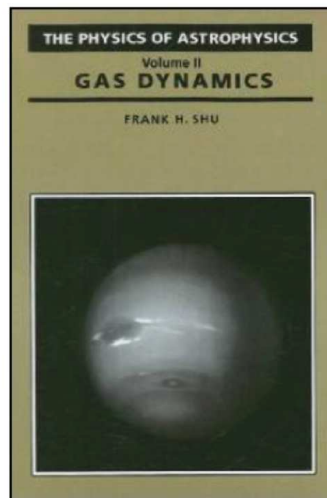
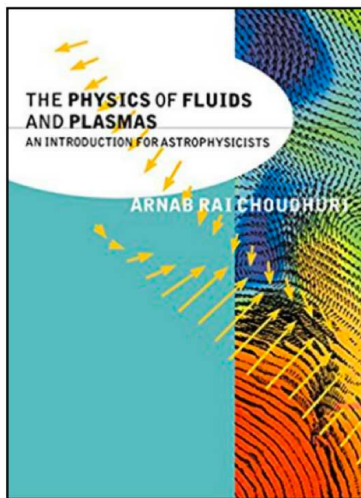
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LITERATURE

The material covered and presented in these lecture notes has relied heavily on a number of excellent textbooks listed below.

- **The Physics of Fluids and Plasmas**
by A. Choudhuri (ISBN-0-521-55543)
- **The Physics of Astrophysics–I. Radiation**
by F. Shu (ISBN-0-935702-64-4)
- **The Physics of Astrophysics–II. Gas Dynamics**
by F. Shu (ISBN-0-935702-65-2)
- **Modern Fluid Dynamics for Physics and Astrophysics**
by O. Regev, O. Umurhan & P. Yecko (ISBN-978-1-4939-3163-7)
- **Principles of Astrophysical Fluid Dynamics**
by C. Clarke & B. Carswell (ISBN-978-0-470-01306-9)
- **Introduction to Modern Magnetohydrodynamics**
by S. Galtier (ISBN-978-1-316-69247-9)
- **Astrophysics: Decoding the Cosmos**
by J. Irwin (ISBN-978-0-470-01306-9)
- **Theoretical Astrophysics**
by M. Bartelmann (ISBN-978-3-527-41004-0)
- **Radiative Processes in Astrophysics**
by G. Rybicki & A. Lightman (ISBN-978-0-471-82759-7)
- **Galactic Dynamics**
by J. Binney & S. Tremaine (ISBN-978-0-691-13027-9)
- **Modern Classical Physics**
by K. Thorne & R. Blandford (ISBN-978-0-691-15902-7)



Part I: Fluid Dynamics



Almost everything we encounter in the Universe, from gas planets to stars, and from the interstellar medium to galaxies, can be categorized as some kind of fluid. Hence, understanding astrophysical processes requires a solid understanding of fluid dynamics. The following chapters present fairly detailed description of the dynamics of fluids with an application to astrophysics.

Fluid dynamics is a rich topic, and one could easily devote an entire course to it. The following chapters therefore only scratch the surface of this rich topic. Readers who want to get more indepth information are referred to the following excellent textbooks

- *The Physics of Fluids and Plasmas* by A. Choudhuri
- *Modern Fluid Dynamics for Physics and Astrophysics* by O.Regev et al.
- *The Physics of Astrophysics II. Gas Dynamics* by F. Shu
- *Principles of Astrophysical Fluid Dynamics* by C. Clarke & B. Carswell
- *Modern Classical Physics* by K.Thorne & R. Blandford

CHAPTER 1

Introduction to Fluids & Plasmas

What is a fluid?

A fluid is a substance that can flow, has no fixed shape, and offers little resistance to an external stress

- In a fluid the constituent particles (atoms, ions, molecules, stars) can ‘freely’ move past one another.
- Fluids take on the shape of their container (if not self-gravitating).
- A fluid changes its shape **at a steady rate** when acted upon by a stress force.

What is a plasma?

A plasma is a fluid in which (some of) the constituent particles are electrically charged, such that the interparticle force (Coulomb force) is long-range in nature.

Fluid Demographics:

All fluids are made up of large numbers of constituent particles, which can be molecules, atoms, ions, dark matter particles or even stars. Different types of fluids mainly differ in the nature of their interparticle forces. Examples of inter-particle forces are the Coulomb force (among charged particles in a plasma), vanderWaals forces (among molecules in a neutral fluid) and gravity (among the stars in a galaxy). Fluids can be both **collisional** or **collisionless**, where we define a collision as *an interaction between constituent particles that causes the trajectory of at least one of these particles to be deflected ‘noticeably’*. Collisions among particles drive the system towards **thermodynamic equilibrium** (at least locally) and the velocity distribution towards a **Maxwell-Boltzmann distribution**.

In neutral fluids the particles only interact with each other on very small scales. Typically the inter-particle force is a vanderWaals force, which drops off very rapidly. Put differently, the typical cross section for interactions is the size of the particles (i.e., the Bohr radius for atoms), which is very small. Hence, to good approximation

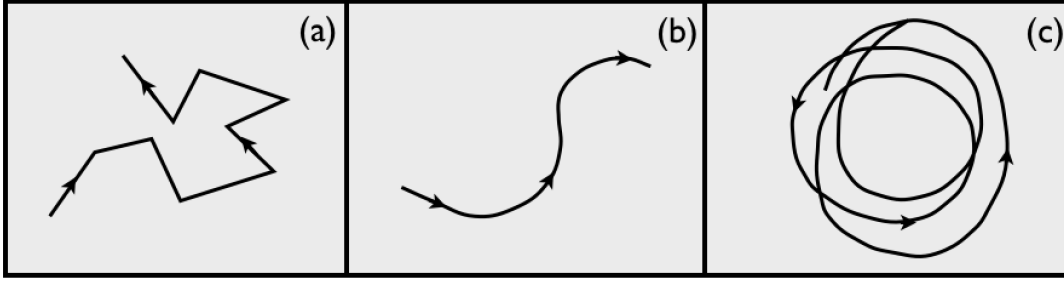


Figure 1: *Examples of particle trajectories in (a) a collisional, neutral fluid, (b) a plasma, and (c) a self-gravitating collisionless, neutral fluid. Note how different the dynamics are.*

particles in a neutral fluid move in straight lines in between highly-localized, large-angle scattering events (**‘collisions’**). An example of such a particle trajectory is shown in Fig. **1a**. Unless the fluid is extremely dilute, most neutral fluids are collisional, meaning that *the mean free path of the particles is short compared to the physical scales of interest*. In astrophysics, though, there are cases where this is not necessarily the case. In such cases, the standard equations of fluid dynamics may not be valid!

In a fully ionized plasma the particles exert Coulomb forces ($\vec{F} \propto r^{-2}$) on each other. Because these are long-range forces, the velocity of a charged particle changes more likely due to a succession of many small deflections rather than due to one large one. As a consequence, particles trajectories in a highly ionized plasma (see Fig. **1b**) are very different from those in a neutral fluid.

In a weakly ionized plasma most interactions/collisions are among neutrals or between neutrals and charged particles. These interactions are short range, and a weakly ionized plasma therefore behaves very much like a neutral fluid.

In astrophysics we often encounter fluids in which the mean, dominant interparticle force is gravity. We shall refer to such fluids as *N*-body systems. Examples are dark matter halos (if dark matter consists of WIMPs or axions) and galaxies (stars act like neutral particles exerting gravitational forces on each other). Since gravity is a long-range force, each particle feels the force from all other particles. Consider the gravitational force $\vec{F}_i$ at a position $\vec{x}_i$ from all particles in a relaxed, equilibrium

system. We can then write that

$$\vec{F}_i(t) = \langle \vec{F} \rangle_i + \delta \vec{F}_i(t)$$

Here $\langle \vec{F} \rangle_i$ is the time (or ensemble) averaged force at the instantaneous position of particle i and $\delta \vec{F}_i(t)$ is the instantaneous deviation due to the discrete nature of the particles that make up the system. As $N \rightarrow \infty$ then $\delta \vec{F}_i \rightarrow 0$ and the system is said to be **collisionless**; *its dynamics are governed by the collective force from all particles rather than by collisions/interactions with individual particles*.

As you learn in Galactic Dynamics, the **relaxation time** of a gravitational N -body system, defined as the time scale on which collisions (i.e., the impact of the $\delta \vec{F}$ above) cause the energies of particles to change considerably, is

$$t_{\text{relax}} \simeq \frac{N}{8 \ln N} t_{\text{cross}}$$

where t_{cross} is the crossing time (comparable to the dynamical time) of the system. Typically $N \sim 10^{10}$ (number of stars in a galaxy) or 10^{50-60} (number of dark matter particles in a halo), and t_{cross} is roughly between 1 and 10 percent of the Hubble time (10^8 to 10^9 yr). Hence, the relaxation time is many times the age of the Universe, and these N -body systems are, for all practical purposes, collisionless. As a consequence, the particle trajectories are (smooth) orbits (see Fig. 1c), and understanding galactic dynamics requires therefore a solid understanding of orbits. Put differently, ‘orbits are the building blocks on galaxies’.

Collisional vs. Collisionless Plasmas: If the collisionality of a gravitational system just depends on N , doesn’t that mean that plasmas are also collisionless? After all, the interparticle force in a plasma is the Coulomb force, which has the same long-range $1/r^2$ nature as gravity. And the number of particles N of a typical plasma is huge ($\gg 10^{10}$) while the dynamical time can be large as well (this obviously depends on the length scales considered, but these tend to be large for astrophysical plasmas).

However, an important difference between a gravitational system and a plasma is that the Coulomb force can be either attractive or repulsive, depending on the electrical charges of the particles. On large scales, plasmas are neutral. This **charge neutrality** is guaranteed by the fact that any charge imbalance would produce strong electrostatic forces that quickly re-establish neutrality. As you will learn in any course on plasma physics (not covered in ASTRO 320), the effect of electrical

charges is screened beyond the **Debye length**:

$$\lambda_D = \left(\frac{k_B T}{8\pi n e^2} \right)^{1/2} \simeq 4.9 \text{ cm } n^{-1/2} T^{1/2}$$

Here n is the number density in cm^{-3} , T is the temperature in degrees Kelvin, and e is the electrical charge of an electron in e.s.u. Related to the Debye length is the **Plasma parameter**

$$g \equiv \frac{1}{n \lambda_D^3} \simeq 8.6 \times 10^{-3} n^{1/2} T^{-3/2}$$

A plasma is (to good approximation) collisionless if the number of particles within the Debye volume, $N_D = n \lambda_D^3 = g^{-1}$ is sufficiently large. After all, only those particles exert Coulomb forces on each other; particles that outside of each others Debye volume do not exert a long-range Coulomb force on each other.

As an example, let's consider three different astrophysical plasmas: the ISM (interstellar medium), the ICM (intra-cluster medium), and the interior of the Sun. The warm phase of the ISM has a temperature of $T \sim 10^4 \text{K}$ and a number density of $n \sim 1 \text{cm}^{-3}$. This implies $N_D \sim 1.2 \times 10^8$. Hence, the warm phase of the ISM can be treated as a collisionless plasma on sufficiently small time-scales (for example when treating high-frequency plasma waves). The ICM has a much lower average density of $\sim 10^{-4} \text{cm}^{-3}$ and a much higher temperature ($\sim 10^7 \text{K}$). This implies a much larger number of particles per Debye volume of $N_D \sim 4 \times 10^{14}$. Hence, the ICM can typically be approximated as a collisionless plasma. The interior of stars, though, has a similar temperature of $\sim 10^7 \text{K}$ but at much higher density ($n \sim 10^{23} \text{cm}^{-3}$), implying $N_D \sim 10$. Hence, stellar interiors are highly collisional plasmas!

Magnetohydrodynamics: Plasma are excellent conductors, and therefore are quickly shorted by currents; hence in many cases one may ignore the electrical field, and focus exclusively on the magnetic field instead. This is called magneto-hydrodynamics, or **MHD** for short. Many astrophysical plasmas have relatively weak magnetic fields, and we therefore don't make big errors if we ignore them. In this case, when electromagnetic interactions are not important, plasmas behave very much like neutral fluids. Because of this, most of the material covered in this course will focus on neutral fluids, despite the fact that more than 99% of all baryonic matter in the Universe is a plasma.

Compressibility: Fluids and plasmas can be either **gaseous** or **liquid**. A gas is **compressible** and will completely fill the volume available to it. A liquid, on the other hand, is (to good approximation) **incompressible**, which means that a liquid of given mass occupies a given volume.

NOTE: Although a gas is said to be compressible, many gaseous *flows* (and virtually all astrophysical flows) are **incompressible**. When the gas is in a container, you can easily compress it with a piston, but if I move my hand (sub-sonically) through the air, the gas adjusts itself to the perturbation in an incompressible fashion (it moves out of the way at the **speed of sound**). The small compression at my hand propagates forward at the speed of sound (**sound wave**) and disperses the gas particles out of the way. In astrophysics we rarely encounter containers, and **subsonic** gas flow is often treated (to good approximation) as being incompressible.

*Throughout what follows, we use ‘fluid’ to mean a **neutral** fluid, and ‘plasma’ to refer to a fluid in which the particles are electrically charged.*

Ideal (Perfect) Fluids and Ideal Gases:

As we discuss in more detail in Chapter 4, the resistance of fluids to shear distortions is called **viscosity**, which is a microscopic property of the fluid that depends on the nature of its constituent particles, and on thermodynamic properties such as temperature. Fluids are also **conductive**, in that the microscopic collisions between the constituent particles cause heat conduction through the fluid. In many fluids encountered in astrophysics, the viscosity and conduction are very small. *An ideal fluid, also called a perfect fluid, is a fluid with zero viscosity and zero conduction.*

NOTE: An ideal (or perfect) fluid should NOT be confused with an ideal or perfect gas, which is defined as a gas in which the pressure is solely due to the kinetic motions of the constituent particles. As we show in Chapter 6, and as you have probably seen before, this implies that the pressure can be written as $P = n k_B T$, with n the particle number density, k_B the Boltzmann constant, and T the temperature.

Examples of Fluids in Astrophysics:

- **Stars:** stars are spheres of gas in **hydrostatic equilibrium** (i.e., gravitational force is balanced by pressure gradients). Densities and temperatures in a given

star cover many orders of magnitude. To good approximation, its **equation of state** is that of an **ideal gas**.

- **Giant (gaseous) planets:** Similar to stars, gaseous planets are large spheres of gas, albeit with a rocky core. Contrary to stars, though, the gas is typically so dense and cold that it can no longer be described with the **equation of state** of an **ideal gas**.
- **Planet atmospheres:** The atmospheres of planets are stratified, gaseous fluids retained by the planet's gravity.
- **White Dwarfs & Neutron stars:** These objects (stellar remnants) can be described as fluids with a **degenerate equation of state**.
- **Proto-planetary disks:** the dense disks of gas and dust surrounding newly formed stars out of which planetary systems form.
- **Inter-Stellar Medium (ISM):** The gas in between the stars in a galaxy. The ISM is typically extremely complicated, and roughly has a **three-phase structure**: it consists of a dense, cold ($\sim 10\text{K}$) molecular phase, a warm ($\sim 10^4\text{K}$) phase, and a dilute, hot ($\sim 10^6\text{K}$) phase. Stars form out of the dense molecular phase, while the hot phase is (shock) heated by supernova explosions. The reason for this three phase medium is associated with the various cooling mechanisms. At high temperature when all gas is ionized, the main cooling channel is Bremsstrahlung (acceleration of free electrons by positively charged ions). At low temperatures ($< 10^4\text{K}$), the main cooling channel is molecular cooling (or cooling through hyperfine transitions in metals).
- **Inter-Galactic Medium (IGM):** The gas in between galaxies. This gas is typically very, very dilute (low density). It is continuously 'exposed' to adiabatic cooling due to the expansion of the Universe, but also is heated by radiation from stars (galaxies) and AGN (active galactic nuclei). The latter, called '**reionization**', assures that the typical temperature of the IGM is $\sim 10^4\text{K}$.
- **Intra-Cluster Medium (ICM):** The hot gas in clusters of galaxies. This is gas that has been shock heated when it fell into the cluster; typically gas passes through an **accretion shock** when it falls into a dark matter halo, converting its infall velocity into thermal motion.

- **Accretion disks:** Accretion disks are gaseous, viscous disks in which the viscosity (enhanced due to turbulence) causes a net rate of radial infall towards the center of the disk, while angular momentum is being transported outwards (accretion)
- **Galaxies (stellar component):** as already mentioned above, the stellar component of galaxies is a collisionless fluid; to very, very good approximation, two stars in a galaxy will never experience a direct collision with another star.
- **Dark matter halos:** Another example of a collisionless fluid (at least, it is often *assumed* that dark matter is collisionless)...

CHAPTER 2

Dynamical Treatments of Fluids

A **dynamical theory** consists of two characteristic elements:

1. a way to describe the **state** of the system
2. a (set of) equation(s) to describe how the state variables change with time

Consider the following examples:

Example 1: a classical dynamical system

This system is described by the position vectors ($\vec{x}$) and momentum vectors ($\vec{p}$) of all the N particles, i.e., by $(\vec{x}_1, \vec{x}_2, \dots, \vec{x}_N, \vec{p}_1, \vec{p}_2, \dots, \vec{p}_N)$.

If the particles are truly classical, in that they can't emit or absorb radiation, then one can define a **Hamiltonian**

$$\mathcal{H}(\vec{x}_i, \vec{p}_i, t) \equiv \mathcal{H}(\vec{x}_1, \vec{x}_2, \dots, \vec{x}_N, \vec{p}_1, \vec{p}_2, \dots, \vec{p}_N, t) = \sum_{i=1}^N \vec{p}_i \cdot \dot{\vec{x}}_i - \mathcal{L}(\vec{x}_i, \dot{\vec{x}}_i, t)$$

where $\mathcal{L}(\vec{x}_i, \dot{\vec{x}}_i, t)$ is the system's **Lagrangian**, and $\dot{\vec{x}}_i = d\vec{x}_i/dt$.

The equations that describe the time-evolution of these state-variables are the Hamiltonian **equations of motion**:

$$\dot{\vec{x}}_i = \frac{\partial \mathcal{H}}{\partial \vec{p}_i}; \quad \dot{\vec{p}}_i = -\frac{\partial \mathcal{H}}{\partial \vec{x}_i}$$

Example 2: an electromagnetic field

The state of this system is described by the electrical and magnetic fields, $\vec{E}(\vec{x})$ and $\vec{B}(\vec{x})$, respectively, and the equations that describe their evolution with time are the **Maxwell equations**, which contain the terms $\partial\vec{E}/\partial t$ and $\partial\vec{B}/\partial t$.

Example 3: a quantum system

The state of a quantum system is fully described by the (complex) **wavefunction** $\psi(\vec{x})$, the time-evolution of which is described by the **Schrödinger equation**

$$i\hbar \frac{\partial\psi}{\partial t} = \hat{\mathcal{H}}\psi$$

where $\hat{\mathcal{H}}$ is now the Hamiltonian operator.

Level	Description of state	Dynamical equations
0: N quantum particles	$\psi(\vec{x}_1, \vec{x}_2, \dots, \vec{x}_N)$	Schrödinger equation
1: N classical particles	$(\vec{x}_1, \vec{x}_2, \dots, \vec{x}_N, \vec{v}_1, \vec{v}_2, \dots, \vec{v}_N)$	Hamiltonian equations
2: Distribution function	$f(\vec{x}, \vec{v}, t)$	Boltzmann equation
3: Continuum model	$\rho(\vec{x}), \vec{u}(\vec{x}), P(\vec{x}), T(\vec{x})$	Hydrodynamic equations

Different levels of dynamical theories to describe neutral fluids

The different levels of Fluid Dynamics

There are different ‘levels’ of dynamical theories to describe fluids. Since all fluids are ultimately made up of constituent particles, and since all particles are ultimately ‘quantum’ in nature, the most ‘basic’ level of fluid dynamics describes the state of a fluid in terms of the N -particle wave function $\psi(\vec{x}_1, \vec{x}_2, \dots, \vec{x}_N)$, which evolves in time according to the Schrödinger equation. We will call this the **level-0** description of fluid dynamics. Since N is typically extremely large, this level-0 description is extremely complicated and utterly unfeasible. Fortunately, it is also unnecessary.

According to what is known as **Ehrenfest’s theorem**, a system of N *quantum* particles can be treated as a system of N *classical* particles if the characteristic separation between the particles is large compared to the ‘de Broglie’ wavelength

$$\lambda = \frac{h}{p} \simeq \frac{h}{\sqrt{mk_{\text{B}}T}}$$

Here h is the Planck constant, p is the particle's momentum, m is the particle mass, k_{B} is the Boltzman constant, and T is the temperature of the fluid. This de Broglie wavelength indicates the ‘characteristic’ size of the wave-packet that according to quantum mechanics describes the particle, and is typically very small. Except for extremely dense fluids such as white dwarfs and neutron stars, or ‘exotic’ types of dark matter (i.e., ‘fuzzy dark matter’), the de Broglie wavelength is always much smaller than the mean particle separation, and classical, Newtonian mechanics suffices. As we have seen above, a classical, Newtonian system of N particles can be described by a Hamiltonian, and the corresponding equations of motions. We refer to this as the **level-1** description of fluid dynamics (see under ‘example 1’ above). Clearly, when N is very large is it unfeasible to solve the $2N$ equations of motion for all the positions and momenta of all particles. We need another approach.

In the **level-2** approach, one introduces the distribution function $f(\vec{x}, \vec{p}, t)$, which describes the number density of particles in 6-dimensional ‘phase-space’ $(\vec{x}, \vec{p})$ (i.e., how many particles are there with positions in the 3D volume $\vec{x} + d\vec{x}$ and momenta in the 3D volume $\vec{p} + d\vec{p}$). The equation that describes how $f(\vec{x}, \vec{p}, t)$ evolves with time is called the **Boltzmann equation** for a neutral fluid. If the fluid is collisionless this reduces to the **Collisionless Boltzmann equation** (CBE). If the collisionless fluid is a plasma, the same equation is called the **Vlasov equation**. Often the CBE and the Vlasov equation are used without distinction.

At the final **level-3**, the fluid is modelled as a continuum. This means we ignore that fluids are made up of constituent particles, and rather describe the fluid with continuous fields, such as the density and velocity fields $\rho(\vec{x})$ and $\vec{u}(\vec{x})$ which assign to each point in space a scalar quantity ρ and a vector quantity $\vec{u}$, respectively. For an ideal neutral fluid, the state in this level-3 approach is fully described by four fields: the density $\rho(\vec{x})$, the velocity field $\vec{u}(\vec{x})$, the pressure $P(\vec{x})$, and the internal, specific energy $\varepsilon(\vec{x})$ (or, equivalently, the temperature $T(\vec{x})$). In the MHD treatment of plasmas one also needs to specify the magnetic field $\vec{B}(\vec{x})$. The equations that describe the time-evolution of $\rho(\vec{x})$, $\vec{u}(\vec{x})$, and $\varepsilon(\vec{x})$ are called the **continuity equation**, the **Navier-Stokes equations**, and the **energy equation**, respectively. Collectively, we shall refer to these as the **hydrodynamic equations** or **fluid equations**. In MHD you have to slightly modify the Navier-Stokes equations, and add an additional **induction equation** describing the time-evolution of the magnetic

field. For an ideal (or perfect) fluid (i.e., no viscosity and/or conductivity), the Navier-Stokes equations reduce to what are known as the **Euler equations**. For a collisionless gravitational system, the equivalent of the Euler equations are called the **Jeans equations**.

Throughout this course, we mainly focus on the level-3 treatment, to which we refer hereafter as the **macroscopic approach**. However, for completeness we will derive these continuum equations starting from a completely general, **microscopic** level-1 treatment. Along the way we will see how subtle differences in the inter-particle forces gives rise to a rich variety in dynamics (fluid vs. plasma, collisional vs. collisionless).

Fluid Dynamics: The Macroscopic Continuum Approach:

In the macroscopic approach, the fluid is treated as a **continuum**. It is often useful to think of this continuum as ‘made up’ of **fluid elements** (FE). These are small fluid volumes that nevertheless contain many particles, that are significantly larger than the mean-free path of the particles, and for which one can define local hydrodynamical variables such as density, pressure and temperature. The requirements are:

1. the FE needs to be much smaller than the characteristic scale in the problem, which is the scale over which the hydrodynamical quantities Q change by an order of magnitude, i.e.

$$l_{\text{FE}} \ll l_{\text{scale}} \sim \frac{Q}{\nabla Q}$$

2. the FE needs to be sufficiently large that fluctuations due to the finite number of particles (‘discreteness noise’) can be neglected, i.e.,

$$n l_{\text{FE}}^3 \gg 1$$

where n is the number density of particles.

3. the FE needs to be sufficiently large that it ‘knows’ about the local conditions through collisions among the constituent particles, i.e.,

$$l_{\text{FE}} \gg \lambda$$

where λ is the mean-free path of the fluid particles.

The ratio of the mean-free path, λ , to the characteristic scale, l_{scale} is known as the **Knudsen number**: $\text{Kn} = \lambda/l_{\text{scale}}$. Fluids typically have $\text{Kn} \ll 1$; if not, then one is not justified in using the continuum approach (level-3) to fluid dynamics, and one is forced to resort to a more statistical approach (level-2).

Note that fluid elements can NOT be defined for a collisionless fluid (which has an infinite mean-free path). This is one of the reasons why one cannot use the macroscopic approach to derive the equations that govern a collisionless fluid.

Fluid Dynamics: closure:

In general, a fluid element is characterized by the following six hydro-dynamical variables:

mass density	ρ	[g/cm ³]	
fluid velocity	$\vec{u}$	[cm/s]	(3 components)
pressure	P	[erg/cm ³]	
specific internal energy	ε	[erg/g]	

Note that $\vec{u}$ is the velocity of the fluid element, not to be confused with the velocity $\vec{v}$ of individual fluid particles, used in the Boltzmann distribution function. Rather, $\vec{u}$ is (roughly) a vector sum of all particles velocities $\vec{v}$ that make up the fluid element.

In the case of an **ideal (or perfect) fluid** (i.e., with zero viscosity and conductivity), the Navier-Stokes equations (which are the hydrodynamical momentum equations) reduce to what are called the **Euler equations**. In that case, the evolution of fluid elements is describe by the following set of hydrodynamical equations:

1 continuum equation	relating ρ and $\vec{u}$
3 momentum equations	relating ρ , $\vec{u}$ and P
1 energy equation	relating ρ , $\vec{u}$, P and ε

Thus we have a total of 5 equations for 6 unknowns. One can solve the set (‘close it’) by using a **constitutive relation**. In almost all cases, this is the **equation of state** (EoS) $P = P(\rho, \varepsilon)$.

- Sometimes the EoS is expressed as $P = P(\rho, T)$. In that case another constitution relation is needed, typically $\varepsilon = \varepsilon(\rho, T)$.

- If the EoS is **barotropic**, i.e., if $P = P(\rho)$, then the energy equation is not needed to close the set of equations. There are two barotropic EoS that are encountered frequently in astrophysics: the **isothermal** EoS, which describes a fluid for which cooling and heating always balance each other to maintain a constant temperature, and the **adiabatic** EoS, in which there is no net heating or cooling (other than adiabatic heating or cooling due to the compression or expansion of volume, i.e., the $P dV$ work). We will discuss these cases in more detail later in the course.
- No EoS exists for a **collisionless fluid**. Consequently, for a collisionless fluid one can never close the set of fluid equations, unless one makes a number of simplifying assumptions (i.e., one postulates various symmetries)
- If the fluid is not ideal, then the momentum equations include terms that contain the (kinetic) **viscosity**, ν , and the energy equation includes a term that contains the conductivity, $\mathcal{K}$. Both ν and $\mathcal{K}$ depend on the mean-free path of the constituent particles and therefore depend on the temperature and collisional cross-section of the particles. Closure of the set of hydrodynamic equations then demands additional constitutive equations $\nu(T)$ and $\mathcal{K}(T)$. Often, though, ν and $\mathcal{K}$ are simply assumed to be constant (the T -dependence is ignored).
- In the case the fluid is exposed to an **external force** (i.e., a gravitational or electrical field), the momentum and energy equations contain an extra force term.
- If the fluid is **self-gravitating** (which is the case, for example, for stars and galaxies) there is an additional unknown, the gravitational potential Φ . However, there is also an additional equation, the **Poisson equation** relating Φ to ρ , so that the set of equations remains closed.
- In the case of a plasma, the charged particles give rise to electric and magnetic fields. Each fluid element now carries 6 additional scalars ($E_x, E_y, E_z, B_x, B_y, B_z$), and the set of equations has to be complemented with the **Maxwell equations** that describe the time evolution of $\vec{E}$ and $\vec{B}$.

Fluid Dynamics: Eulerian *vs.* Lagrangian Formalism:

One distinguishes two different formalisms for treating fluid dynamics:

- **Eulerian Formalism:** in this formalism one solves the fluid equations ‘at fixed positions’: the evolution of a quantity Q is described by the local (or partial, or Eulerian) derivative $\partial Q/\partial t$. An Eulerian hydrodynamics code is a ‘grid-based code’, which solves the hydro equations on a fixed grid, or using an adaptive grid, which refines resolution where needed. The latter is called **Adaptive Mesh Refinement** (AMR).
- **Lagrangian Formalism:** in this formalism one solves the fluid equations ‘comoving with the fluid’, i.e., either at a fixed particle (collisionless fluid) or at a fixed fluid element (collisional fluid). The evolution of a quantity Q is described by the substantial (or Lagrangian) derivative dQ/dt (sometimes written as DQ/Dt). A Lagrangian hydrodynamics code is a ‘particle-based code’, which solves the hydro equations per simulation particle. Since it needs to smooth over neighboring particles in order to compute quantities such as the fluid density, it is called **Smoothed Particle Hydrodynamics** (SPH).

To derive an expression for the **substantial derivative** dQ/dt , realize that $Q = Q(t, x, y, z)$. When the fluid element moves, the scalar quantity Q experiences a change

$$dQ = \frac{\partial Q}{\partial t} dt + \frac{\partial Q}{\partial x} dx + \frac{\partial Q}{\partial y} dy + \frac{\partial Q}{\partial z} dz$$

Dividing by dt yields

$$\frac{dQ}{dt} = \frac{\partial Q}{\partial t} + \frac{\partial Q}{\partial x} u_x + \frac{\partial Q}{\partial y} u_y + \frac{\partial Q}{\partial z} u_z$$

where we have used that $dx/dt = u_x$, which is the x -component of the fluid velocity $\vec{u}$, etc. Hence we have that

$$\boxed{\boxed{\frac{dQ}{dt} = \frac{\partial Q}{\partial t} + \vec{u} \cdot \nabla Q}}$$

Using a similar derivation, but now for a vector quantity $\vec{A}(\vec{x}, t)$, it is straightforward to show that

$$\boxed{\frac{d\vec{A}}{dt} = \frac{\partial \vec{A}}{\partial t} + (\vec{u} \cdot \nabla) \vec{A}}$$

which, in index-notation, is written as

$$\frac{dA_i}{dt} = \frac{\partial A_i}{\partial t} + u_j \frac{\partial A_i}{\partial x_j}$$

Another way to derive the above relation between the Eulerian and Lagrangian derivatives, is to think of dQ/dt as

$$\frac{dQ}{dt} = \lim_{\delta t \rightarrow 0} \left[\frac{Q(\vec{x} + \delta \vec{x}, t + \delta t) - Q(\vec{x}, t)}{\delta t} \right]$$

Using that

$$\vec{u} = \lim_{\delta t \rightarrow 0} \left[\frac{\vec{x}(t + \delta t) - \vec{x}(t)}{\delta t} \right] = \frac{\delta \vec{x}}{\delta t}$$

and

$$\nabla Q = \lim_{\delta \vec{x} \rightarrow 0} \left[\frac{Q(\vec{x} + \delta \vec{x}, t) - Q(\vec{x}, t)}{\delta \vec{x}} \right]$$

it is straightforward to show that this results in the same expression for the substantial derivative as above.

Kinematic Concepts: Streamlines, Streaklines and Particle Paths:

In fluid dynamics it is often useful to distinguish the following kinematic constructs:

- **Streamlines:** curves that are instantaneously tangent to the velocity vector of the flow. Streamlines show the direction a massless fluid element will travel in at any point in time.
- **Streaklines:** the locus of points of all the fluid particles that have passed continuously through a particular spatial point in the past. Dye steadily injected into the fluid at a fixed point extends along a streakline.

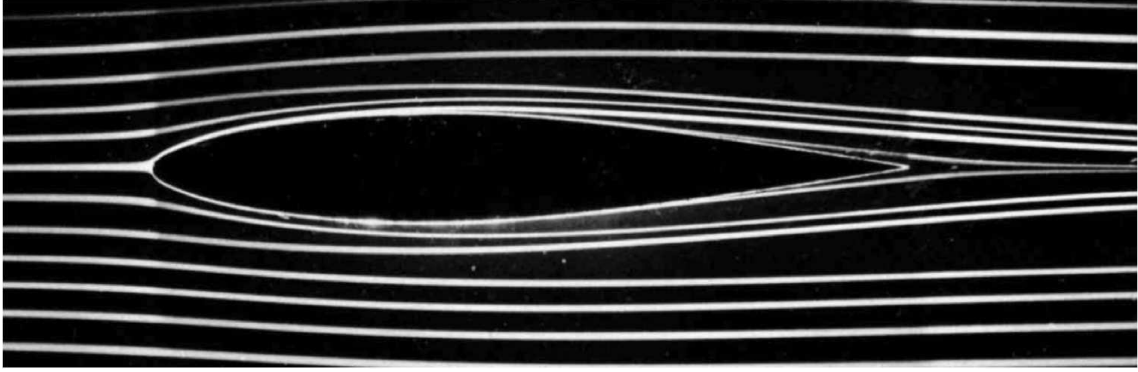


Figure 2: *Streaklines showing laminar flow across an airfoil; made by injecting dye at regular intervals in the flow*

- **Particle paths:** (aka pathlines) are the trajectories that individual fluid elements follow. The direction the path takes is determined by the streamlines of the fluid at each moment in time.

Only if the flow is **steady**, which means that all partial time derivatives (i.e., $\partial \vec{u}/\partial t = \partial \rho/\partial t = \partial P/\partial t$) vanish, will streamlines be identical to streaklines be identical to particle paths. For a non-steady flow, they will differ from each other.

CHAPTER 3

Hydrodynamic Equations for Ideal Fluid

Without any formal derivation (this comes later) we now present the hydrodynamic equations for an ideal, neutral fluid. Note that these equations adopt the level-3 continuum approach discussed in the previous chapter.

	Lagrangian	Eulerian
Continuity Eq:	$\frac{d\rho}{dt} = -\rho \nabla \cdot \vec{u}$	$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \vec{u}) = 0$
Momentum Eqs:	$\frac{d\vec{u}}{dt} = -\frac{\nabla P}{\rho} - \nabla \Phi$	$\frac{\partial \vec{u}}{\partial t} + (\vec{u} \cdot \nabla) \vec{u} = -\frac{\nabla P}{\rho} - \nabla \Phi$
Energy Eq:	$\frac{d\varepsilon}{dt} = -\frac{P}{\rho} \nabla \cdot \vec{u} - \frac{\mathcal{L}}{\rho}$	$\frac{\partial \varepsilon}{\partial t} + \vec{u} \cdot \nabla \varepsilon = -\frac{P}{\rho} \nabla \cdot \vec{u} - \frac{\mathcal{L}}{\rho}$

Hydrodynamic equations for an ideal, neutral fluid in gravitational field

NOTE: students should become familiar with switching between the **Eulerian** and **Lagrangian** equations, and between the **vector notation** shown above and the **index notation**. The latter is often easier to work with. When writing down the index versions, make sure that each term carries the same index, and make use of the Einstein summation convention. The only somewhat tricky term is the $(\vec{u} \cdot \nabla) \vec{u}$ -term in the Eulerian momentum equations, which in index form is given by $u_j(\partial u_i / \partial x_j)$, where i is the index carried by each term of the equation.

Continuity Equation: this equation expresses mass conservation. This is clear from the Eulerian form, which shows that changing the density at some fixed point in space requires a converging, or diverging, mass flux at that location.

If a flow is incompressible, then $\nabla \cdot \vec{u} = 0$ everywhere, and we thus have that $d\rho/dt = 0$ (i.e., the density of each fluid element is fixed in time as it moves with the flow). If a fluid is incompressible, then $d\rho/dt = 0$ and we see that the flow is divergence free

$(\nabla \cdot \vec{u} = 0)$, which is also called **solenoidal**.

Momentum Equations: these equations simply state that one can accelerate a fluid element with either a gradient in the pressure, P , or a gradient in the gravitational potential, Φ . Basically these momentum equations are nothing but Newton's $\vec{F} = m\vec{a}$ applied to a fluid element. In the above form, valid for an inviscid, ideal fluid, the momentum equations are called the **Euler equations**.

Energy Equation: the energy equation states that the only way that the **specific, internal energy**, ε , of a fluid element can change, in the absence of conduction, is by adiabatic compression or expansion, which requires a non-zero divergence of the velocity field (i.e., $\nabla \cdot \vec{u} \neq 0$), or by radiation (emission or absorption of photons). The latter is expressed via the **net volumetric cooling rate**,

$$\mathcal{L} = \rho \frac{dQ}{dt} = \mathcal{C} - \mathcal{H}$$

Here Q is the thermodynamic heat, and $\mathcal{C}$ and $\mathcal{H}$ are the net volumetric cooling and heating rates, respectively.

If the ideal fluid is governed by self-gravity (as opposed to, is placed in an external gravitational field), then one needs to complement the hydrodynamical equations with the **Poisson equation**: $\nabla^2 \Phi = 4\pi G \rho$. In addition, closure requires an additional **constitutive relations** in the form of an equation-of-state $P = P(\rho, \varepsilon)$. If the ideal fluid obeys the ideal gas law, then we have the following two constitutive relations:

$$P = \frac{k_B T}{\mu m_p} \rho, \quad \varepsilon = \frac{1}{\gamma - 1} \frac{k_B T}{\mu m_p}$$

(see Chapter 6 for details). Here μ is the mean molecular weight of the fluid in units of the proton mass, m_p , and γ is the adiabatic index, which is often taken to be 5/3 as appropriate for a mono-atomic gas.

Especially for the numerical Eulerian treatment of fluids, it is advantageous to write the hydro equations in **conservative form**. Let $A(\vec{x}, t)$ be some state variable of the fluid (either scalar or vector). The evolution equation for A is said to be in conservative form if

$$\frac{\partial A}{\partial t} + \nabla \cdot \vec{F}(A) = S$$

Here $\vec{F}(A)$ describes the appropriate flux of A and S describes the various sources and/or sinks of A . The continuity, momentum and energy equations for an ideal fluid in conservative form are:

$$\begin{aligned}\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \vec{u}) &= 0 \\ \frac{\partial \rho \vec{u}}{\partial t} + \nabla \cdot \overline{\overline{\Pi}} &= -\rho \nabla \Phi \\ \frac{\partial E}{\partial t} + \nabla \cdot [(E + P) \vec{u}] &= \rho \frac{\partial \Phi}{\partial t} - \mathcal{L}\end{aligned}$$

Here

$$\overline{\overline{\Pi}} = \rho \vec{u} \otimes \vec{u} + P$$

is the **momentum flux density tensor** (of rank 2), and

$$E = \rho \left(\frac{1}{2} u^2 + \Phi + \varepsilon \right)$$

is the **energy density**.

NOTE: In the expression for the momentum flux density tensor $\vec{A} \otimes \vec{B}$ is the **tensor product** of $\vec{A}$ and $\vec{B}$ defined such that $(\vec{A} \otimes \vec{B})_{ij} = a_i b_j$ (see Appendix A). Hence, the index-form of the momentum flux density tensor is simply $\Pi_{ij} = \rho u_i u_j + P \delta_{ij}$, with δ_{ij} the Kronecker delta function. Note that this expression is ONLY valid for an ideal fluid; in the next chapter we shall derive a more general expression for the momentum flux density tensor.

Note also that whereas there is no source or sink term for the density, gradients in the gravitational field act as a source of momentum, while its time-variability can cause an increase or decrease in the energy density of the fluid (if the fluid is collisionless, we call this **violent relaxation**). Another source/sink term for the energy density is radiation (emission or absorption of photons).

CHAPTER 4

Viscosity, Conductivity & The Stress Tensor

The hydrodynamic equations presented in the previous chapter are only valid for an **ideal fluid**, i.e., a fluid without viscosity and conduction. We now examine the origin of **conduction** and **viscosity**, and link the latter to the stress tensor, which is an important quantity in all of fluid dynamics.

In an ideal fluid, the particles effectively have a mean-free path of zero, such that they cannot communicate with their neighboring particles. In reality, though, the mean-free path, $\lambda_{\text{mfp}} = (n\sigma)^{-1}$ is finite, and particles "communicate" with each other through collisions. These collisions cause an exchange of momentum and energy among the particles involved, acting as a relaxation mechanism. Note that in a collisionless system the mean-free path is effectively infinite, and there is no two-body relaxation, only collective relaxation mechanisms (i.e., violent relaxation or wave-particle interactions).

- When there are gradients in velocity ("**shear**") then the collisions among neighboring fluid elements give rise to a **net transport of momentum**. The collisions drive the system towards equilibrium, i.e., towards no shear. Hence, the collisions act as a resistance to shear, which is called **viscosity**. See Fig. 3 for an illustration.
- When there are gradients in temperature (or, in other words, in specific internal energy), then the collisions give rise to a **net transport of energy**. Again, the collisions drive the system towards equilibrium, in which the gradients vanish, and the rate at which the fluid can erase a non-zero ∇T is called the **(thermal) conductivity**.

The viscosity, μ , and conductivity, $\mathcal{K}$, are called **transport coefficients**. Expressions for μ and $\mathcal{K}$ in terms of the collision cross section, σ , the fluid's temperature T , and the particle mass m , can be derived in a rigorous manner using what is known as the **Chapman-Enskog expansion**. This is a fairly complicated topic, that is outside of the scope of this course. Interested reader should consult the classical monographs "*Statistical Mechanics*" by K. Huang, or "*The Mathematical Theory of*

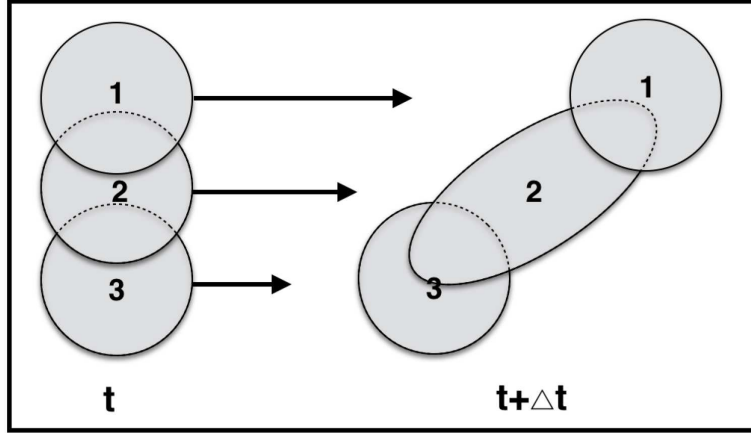


Figure 3: *Illustration of origin of viscosity and shear stress. Three neighboring fluids elements (1, 2 and 3) have different streaming velocities, $\vec{u}$. Due to the microscopic motions and collisions (characterized by a non-zero mean free path), there is a net **transfer of momentum** from the faster moving fluid elements to the slower moving fluid elements. This net transfer of momentum will tend to erase the shear in $\vec{u}(\vec{x})$, and therefore manifests itself as a shear-resistance, known as **viscosity**. Due to the transfer of momentum, the fluid elements deform; in our figure, 1 transfers linear momentum to the top of 2, while 3 extracts linear momentum from the bottom of 2. Consequently, fluid element 2 is sheared as depicted in the figure at time $t + \Delta t$. From the perspective of fluid element 2, some internal force (from within its boundaries) has exerted a **shear-stress** on its bounding surface.*

Non-uniform Gases” by S. Chapman and T. Cowling. Using the Chapman-Enskog expansion one finds the following expressions for μ and $\mathcal{K}$:

$$\mu = \frac{a}{\sigma} \left(\frac{m k_B T}{\pi} \right)^{1/2}, \quad \mathcal{K} = \frac{5}{2} c_V \mu$$

Here a is a numerical factor that depends on the details of the interparticle forces, σ is the collisional cross section, and c_V is the specific heat (i.e., per unit mass). Thus, for a given fluid (given σ and m) we basically have that $\mu = \mu(T)$ and $\mathcal{K} = \mathcal{K}(T)$.

Note that $\mu \propto T^{1/2}$; viscosity increases with temperature. This only holds for gases! For liquids we know from experience that viscosity *decreases* with increasing temperature (think of honey). Since in astrophysics we are mainly concerned with gas, $\mu \propto T^{1/2}$ will be a good approximation for most of what follows.

Now that we have a rough idea of what viscosity (resistance to shear) and conductivity (resistance to temperature gradients) are, we have to ask how to incorporate them into our **hydrodynamic equations**.

Both transport mechanisms relate to the microscopic velocities of the individual particles. However, the macroscopic continuum approach of fluid dynamics only deals with the **streaming velocities** $\vec{u}$, which represents the velocities of the fluid elements. In order to link these different velocities we proceed as follows:

Velocity of fluid particles: We split the velocity, $\vec{v}$, of a fluid particle in a streaming velocity, $\vec{u}$, and a ‘random’ velocity, $\vec{w}$:

$$\vec{v} = \vec{u} + \vec{w}$$

where $\langle \vec{v} \rangle = \vec{u}$, $\langle \vec{w} \rangle = 0$ and $\langle . \rangle$ indicates the average over a fluid *element*. If we define v_i as the velocity in the i -direction, we have that

$$\langle v_i v_j \rangle = u_i u_j + \langle w_i w_j \rangle$$

These different velocities allow us to define a number of different velocity tensors:

Stress Tensor:	$\sigma_{ij} \equiv -\rho \langle w_i w_j \rangle$	$\bar{\bar{\sigma}} = -\rho \vec{w} \otimes \vec{w}$
Momentum Flux Density Tensor:	$\Pi_{ij} \equiv +\rho \langle v_i v_j \rangle$	$\bar{\bar{\Pi}} = +\rho \vec{v} \otimes \vec{v}$
Ram Pressure Tensor:	$\Sigma_{ij} \equiv +\rho u_i u_j$	$\bar{\bar{\Sigma}} = +\rho \vec{u} \otimes \vec{u}$

which are related according to $\overline{\overline{\sigma}} = \overline{\overline{\Sigma}} - \overline{\overline{\Pi}}$. Note that each of these tensors is manifest **symmetric** (i.e., $\sigma_{ij} = \sigma_{ji}$, etc.), which implies that they have 6 independent variables.

Note that the **stress tensor** is related to the microscopic random motions. These are the ones that give rise to pressure, viscosity and conductivity! The reason that σ_{ij} is called the stress tensor is that it is related to the **stress** $\vec{\Sigma}(\vec{x}, \hat{n})$ acting on a surface with normal vector $\hat{n}$ located at $\vec{x}$ according to

$$\Sigma_i(\hat{n}) = \sigma_{ij} n_j$$

Here $\Sigma_i(\hat{n})$ is the i -component of the stress acting on a surface with normal $\hat{n}$, whose j -component is given by n_j . Hence, in general the stress will not necessarily be along the normal to the surface, and it is useful to decompose the stress in a **normal stress**, which is the component of the stress along the normal to the surface, and a **shear stress**, which is the component along the tangent to the surface.

To see that fluid elements in general are subjected to shear stress, consider the following: Consider a flow (i.e., a river) in which we inject a small, spherical blob (a fluid element) of dye. If the only stress to which the blob is subject is normal stress, the only thing that can happen to the blob is an overall compression or expansion. However, from experience we know that the blob of dye will shear into an extended, ‘spaghetti’-like feature; hence, the blob is clearly subjected to shear stress, and this shear stress is obviously related to another tensor called the **deformation tensor**

$$T_{ij} = \frac{\partial u_i}{\partial x_j}$$

which describes the (local) shear in the fluid flow.

Since $\partial u_i / \partial x_j = 0$ in a **static** fluid ($\vec{u}(\vec{x}) = 0$), we see that in a static fluid the stress tensor can only depend on the normal stress, which we call the **pressure**.

Pascal’s law for hydrostatics: In a **static** fluid, there is no preferred direction, and hence the (normal) stress has to be isotropic:

$$\text{static fluid} \quad \Longleftrightarrow \quad \sigma_{ij} = -P \delta_{ij}$$

The minus sign is a consequence of the sign convention of the stress.

Sign Convention: The stress $\vec{\Sigma}(\vec{x}, \hat{n})$ acting at location $\vec{x}$ on a surface with normal $\hat{n}$, is exerted *by* the fluid on the side of the surface to which the normal points, *on* the fluid from which the normal points. In other words, *a positive stress results in compression*. Hence, in the case of pure, normal pressure, we have that $\Sigma = -P$.

Viscous Stress Tensor: The expression for the stress tensor in the case of static fluid motivates us to write in general

$$\sigma_{ij} = -P\delta_{ij} + \tau_{ij}$$

where we have introduced a new tensor, τ_{ij} , which is known as the viscous stress tensor, or the **deviatoric stress tensor**.

Since the deviatoric stress tensor, τ_{ij} , is only non-zero in the presence of shear in the fluid flow, this suggests that

$$\tau_{ij} = T_{ijkl} \frac{\partial u_k}{\partial x_l}$$

where T_{ijkl} is a proportionality tensor of **rank** four. As described in Appendix F (which is NOT part of the curriculum for this course), most (astrophysical) fluids are **Newtonian**, in that they obey a number of conditions. As detailed in that appendix, for a Newtonian fluid, the relation between the stress tensor and the deformation tensor is given by

$$\sigma_{ij} = -P\delta_{ij} + \mu \left[\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} - \frac{2}{3}\delta_{ij} \frac{\partial u_k}{\partial x_k} \right] + \eta \delta_{ij} \frac{\partial u_k}{\partial x_k}$$

Here P is the pressure, δ_{ij} is the Kronecker delta function, μ is the **coefficient of shear viscosity**, and η is the **coefficient of bulk viscosity** (aka the ‘second viscosity’). We thus see that for a Newtonian fluid, the stress tensor, despite being a symmetric tensor of rank two (which implies 6 independent variables), only has three independent components: P , μ and η .

Let’s take a closer look at these three quantities, starting with the pressure P . To be exact, P is the **thermodynamic equilibrium pressure**, and is normally computed thermodynamically from some equation of state, $P = P(\rho, T)$. It is related to the

translational kinetic energy of the particles when the fluid, in equilibrium, has reached **equipartition** of energy among **all** its degrees of freedom, including (in the case of molecules) rotational and vibrations degrees of freedom.

In addition to the thermodynamic equilibrium pressure, P , we can also define a **mechanical pressure**, P_m , which is purely related to the translational motion of the particles, **independent** of whether the system has reached full equipartition of energy. The mechanical pressure is simply the average normal stress and therefore follows from the stress tensor according to

$$P_m = -\frac{1}{3} \text{Tr}(\sigma_{ij}) = -\frac{1}{3} (\sigma_{11} + \sigma_{22} + \sigma_{33})$$

Using the above expression for σ_{ij} , and using that $\partial u_k / \partial x_k = \nabla \cdot \vec{u}$ (Einstein summation convention), it is easy to see that

$$P_m = P - \eta \nabla \cdot \vec{u}$$

From this expression it is clear that the bulk viscosity, η , is only non-zero if $P \neq P_m$. This, in turn, can only happen if the constituent particles of the fluid have degrees of freedom beyond position and momentum (i.e., when they are molecules with rotational or vibrational degrees of freedom). Hence, for a fluid of monoatoms (ideal gas), $\eta = 0$. From the fact that $P = P_m + \eta \nabla \cdot \vec{u}$ it is clear that for an **incompressible flow** $P = P_m$ and the value of η is irrelevant; *bulk viscosity plays no role in incompressible fluids or flows*. The only time when $P_m \neq P$ is when a fluid consisting of particles with internal degrees of freedom (e.g., molecules) has just undergone a large volumetric change (i.e., during a shock). In that case there may be a lag between the time the translational motions reach equilibrium and the time when the system reaches full equipartition in energy among all degrees of freedom. In astrophysics, bulk viscosity can generally be ignored, but be aware that it may be important in shocks. This only leaves the shear viscosity μ , which describes the ability of the fluid to resist shear stress via momentum transport resulting from collisions and the non-zero mean free path of the particles.

CHAPTER 5

Hydrodynamic Equations for Non-Ideal Fluid

In the hydrodynamic equations for an ideal fluid presented in Chapter 3 we ignored both viscosity and conductivity. We now examine how our hydrodynamic equations change when allowing for these two transport mechanisms.

As we have seen in the previous chapter, the effect of **viscosity** is captured by the **stress tensor**, which is given by

$$\sigma_{ij} = -P\delta_{ij} + \tau_{ij} = -P\delta_{ij} + \mu \left[\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} - \frac{2}{3}\delta_{ij} \frac{\partial u_k}{\partial x_k} \right] + \eta \delta_{ij} \frac{\partial u_k}{\partial x_k}$$

Note that in the limit $\mu \rightarrow 0$ and $\eta \rightarrow 0$, valid for an ideal fluid, $\sigma_{ij} = -P\delta_{ij}$. This suggests that we can incorporate viscosity in the hydrodynamic equations by simply replacing the pressure P with the stress tensor, i.e., $P\delta_{ij} \rightarrow -\sigma_{ij} = P\delta_{ij} - \tau_{ij}$.

Starting from the **Euler equation** in Lagrangian index form;

$$\rho \frac{du_i}{dt} = -\frac{\partial P}{\partial x_i} - \rho \frac{\partial \Phi}{\partial x_i}$$

we use that $\partial P / \partial x_i = \partial(P\delta_{ij}) / \partial x_j$, and then make the above substitution to obtain

$$\boxed{\rho \frac{du_i}{dt} = \frac{\partial(-P\delta_{ij})}{\partial x_j} + \frac{\partial \tau_{ij}}{\partial x_j} - \rho \frac{\partial \Phi}{\partial x_i}}$$

These momentum equations are called the **Navier-Stokes equations**.

It is more common, and more useful, to write out the viscous stress tensor, yielding

$$\boxed{\rho \frac{du_i}{dt} = -\frac{\partial P}{\partial x_i} + \frac{\partial}{\partial x_j} \left[\mu \left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} - \frac{2}{3}\delta_{ij} \frac{\partial u_k}{\partial x_k} \right) \right] + \frac{\partial}{\partial x_i} \left(\eta \frac{\partial u_k}{\partial x_k} \right) - \rho \frac{\partial \Phi}{\partial x_i}}$$

These are the **Navier-Stokes** equations (in Lagrangian index form) in all their glory, containing both the **shear viscosity** term and the **bulk viscosity** term (the latter is often ignored).

Note that μ and η are usually functions of density and temperature so that they have spatial variations. However, it is common to assume that these are sufficiently small so that μ and η can be treated as constants, in which case they can be taken outside the differentials. In what follows we will make this assumption as well.

The Navier-Stokes equations in **Lagrangian vector form** are

$$\rho \frac{d\vec{u}}{dt} = -\nabla P + \mu \nabla^2 \vec{u} + \left(\eta + \frac{1}{3}\mu \right) \nabla(\nabla \cdot \vec{u}) - \rho \nabla \Phi$$

If we ignore the bulk viscosity ($\eta = 0$) then this reduces to

$$\boxed{\boxed{\frac{d\vec{u}}{dt} = -\frac{\nabla P}{\rho} + \nu \left[\nabla^2 \vec{u} + \frac{1}{3} \nabla(\nabla \cdot \vec{u}) \right] - \nabla \Phi}}$$

where we have introduced the **kinetic viscosity** $\nu \equiv \mu/\rho$. Note that these equations reduce to the **Euler equations** in the limit $\nu \rightarrow 0$. Also, note that the $\nabla(\nabla \cdot \vec{u})$ term is only significant in the case of flows with **variable compression** (i.e., viscous dissipation of acoustic waves or shocks), and can often be ignored. This leaves the $\nu \nabla^2 \vec{u}$ term as the main addition to the Euler equations. Yet, this simple ‘diffuse’ term (describing **viscous momentum diffusion**) dramatically changes the character of the equation, as it introduces a higher spatial derivative. Hence, additional boundary conditions are required to solve the equations. When solving problems with solid boundaries (not common in astrophysics), this condition is typically that the *tangential* (or shear) velocity at the boundary vanishes. Although this may sound ad hoc, it is supported by observation; for example, the blades of a fan collect dust.

Recall that when writing the Navier-Stokes equation in **Eulerian** form, we have that $d\vec{u}/dt \rightarrow \partial\vec{u}/\partial t + \vec{u} \cdot \nabla \vec{u}$. It is often useful to rewrite this extra term using the vector calculus identity

$$\vec{u} \cdot \nabla \vec{u} = \nabla \left(\frac{\vec{u} \cdot \vec{u}}{2} \right) + (\nabla \times \vec{u}) \times \vec{u}$$

Hence, for an **irrotational flow** (i.e., a flow for which $\nabla \times \vec{u} = 0$), we have that $\vec{u} \cdot \nabla \vec{u} = \frac{1}{2} \nabla u^2$, where $u \equiv |\vec{u}|$.

Next we move to the **energy equation**, modifying it so as to account for both **viscosity** and **conduction**. We start from

$$\rho \frac{d\varepsilon}{dt} = -P \frac{\partial u_i}{\partial x_i} - \mathcal{L}$$

(see Chapter 3, and recall that, with the Einstein summation convention, $\partial u_i / \partial x_i = \nabla \cdot \vec{u}$). As with the momentum equations, we include viscosity by making the transformation $-P\delta_{ij} \rightarrow \sigma_{ij} = -P\delta_{ij} + \tau_{ij}$, which we do as follows:

$$-P \frac{\partial u_i}{\partial x_i} \rightarrow -P\delta_{ij} \frac{\partial u_i}{\partial x_j} \rightarrow -P\delta_{ij} \frac{\partial u_i}{\partial x_j} + \tau_{ij} \frac{\partial u_i}{\partial x_j}$$

This allows us to write the energy equation in vector form as

$$\rho \frac{d\varepsilon}{dt} = -P \nabla \cdot \vec{u} + \mathcal{V} - \mathcal{L}$$

where

$$\mathcal{V} \equiv \tau_{ik} \frac{\partial u_i}{\partial x_k}$$

is the **rate of viscous dissipation** which describes the rate at which the work done against viscous forces is irreversibly converted into internal energy.

Now that we have added the effect of viscosity, what remains is to add **conduction**. We can make progress by realizing that, on the microscopic level, conduction arises from collisions among the constituent particles, causing a flux in internal energy. The internal energy density of a fluid element is $\langle \frac{1}{2} \rho w^2 \rangle$, where $\vec{w} = \vec{v} - \vec{u}$ is the random motion of the particle wrt the fluid element (see Chapter 4), and the angle brackets indicate an ensemble average over the particles that make up the fluid element. Based on this we see that the **conductive flux** in the i -direction can be written as

$$F_{\text{cond},i} = \langle \frac{1}{2} \rho w^2 w_i \rangle = \langle \rho \varepsilon w_i \rangle$$

From experience we also know that we can write the conductive flux as

$$\vec{F}_{\text{cond}} = -\mathcal{K} \nabla T$$

with $\mathcal{K}$ the **thermal conductivity**.

Next we realize that conduction only causes a *net* change in the internal energy at some fixed position if the divergence in the conductive flux ($\nabla \cdot \vec{F}_{\text{cond}}$) at that position is non-zero. This suggests that the final form of the energy equation, for a non-ideal fluid, and in Lagrangian vector form, has to be

$$\boxed{\rho \frac{d\varepsilon}{dt} = -P \nabla \cdot \vec{u} - \nabla \cdot \vec{F}_{\text{cond}} + \mathcal{V} - \mathcal{L}}$$

To summarize, below we list the full set of equations of gravitational, radial hydrodynamics (ignoring bulk viscosity) ².

Full Set of Equations of Gravitational, Radiative Hydrodynamics

Continuity Eq.	$\frac{d\rho}{dt} = -\rho \nabla \cdot \vec{u}$
Momentum Eqs.	$\rho \frac{d\vec{u}}{dt} = -\nabla P + \mu \left[\nabla^2 \vec{u} + \frac{1}{3} \nabla (\nabla \cdot \vec{u}) \right] - \rho \nabla \Phi$
Energy Eq.	$\rho \frac{d\varepsilon}{dt} = -P \nabla \cdot \vec{u} - \nabla \cdot \vec{F}_{\text{cond}} - \mathcal{L} + \mathcal{V}$
Poisson Eq.	$\nabla^2 \Phi = 4\pi G \rho$
Constitutive Eqs.	$P = P(\rho, \varepsilon), \quad \mu = \mu(T) \propto \frac{1}{\sigma} \left(\frac{mk_{\text{B}}T}{\pi} \right)^{1/2}, \quad \mathcal{K} = \mathcal{K}(T) \simeq \frac{5}{2} \mu(T) c_{\text{V}}$
Diss/Cond/Rad	$\mathcal{V} \equiv \tau_{ik} \frac{\partial u_i}{\partial x_k}, \quad F_{\text{cond},k} = \langle \rho \varepsilon w_k \rangle, \quad \mathcal{L} \equiv \mathcal{C} - \mathcal{H}$

²Diss/Cond/Rad stands for Dissipation, Conduction, Radiation

CHAPTER 6

Equations of State

Equation of State (EoS): a thermodynamic equation describing the state of matter under a given set of physical conditions. In what follows we will always write our EoS in the form $P = P(\rho, T)$. Other commonly used forms are $P = P(\rho, \varepsilon)$ or $P = P(\rho, S)$. In the latter, S is the entropy.

Closure: The hydrodynamic equations for an ideal fluid (continuity eq, momentum eqs, and energy eq) are 5 equations with 7 unknowns: $(\rho, \vec{u}, P, T \text{ (or } \varepsilon), \text{ and } \Phi)$. With the addition of the **Poisson equation**, which relates ρ and Φ . The seventh and final equation that ensures closure is the **equation of state** $P = P(\rho, T)$. Note that if the EoS is **barotropic**, i.e., $P = P(\rho)$, then the continuity, momentum and Poisson equations form a closed set, and the energy equation is not required.

Ideal Gas: a hypothetical gas that consists of identical point particles (i.e. of zero volume) that undergo perfectly elastic collisions and for which interparticle forces can be neglected.

An ideal gas obeys the **ideal gas law**: $PV = N k_B T$.

Here N is the total number of particles, k_B is Boltzmann's constant, and V is the volume occupied by the fluid. Using that $\rho = N \mu m_p / V$, where μ is the **mean molecular weight** in units of the proton mass m_p , we have that the **EoS for an ideal gas** is given by

$$P = P(\rho, T) = \frac{k_B T}{\mu m_p} \rho$$

NOTE: astrophysical gases are often well described by the ideal gas law. Even for a fully ionized gas, the interparticle forces (Coulomb force) can typically be neglected (i.e., the potential energies involved are typically $< 10\%$ of the kinetic energies). Ideal gas law breaks down for dense, and cool gases, such as those present in gaseous planets.

Maxwell-Boltzmann Distribution: the distribution of particle momenta, $\vec{p} = m\vec{v}$, of an ideal gas follows the Maxwell-Boltzmann distribution.

$$\mathcal{P}(\vec{p}) d^3\vec{p} = \left(\frac{1}{2\pi m k_B T} \right)^{3/2} \exp \left(-\frac{p^2}{2m k_B T} \right) d^3\vec{p}$$

where $p^2 = \vec{p} \cdot \vec{p}$. This distribution follows from maximizing entropy under the following assumptions:

1. all magnitudes of velocity are *a priori* equally likely
2. all directions are equally likely (isotropy)
3. total energy is constrained at a fixed value
4. total number of particles is constrained at a fixed value

Using that $E = p^2/2m$ we thus see that $\mathcal{P}(\vec{p}) \propto e^{-E/k_B T}$.

NOTE: if there are temperature gradients in the gas, then the particle momenta only follow the Maxwell-Boltzmann distribution **locally**, with $T = T(\vec{x})$ being the local temperature.

Pressure: pressure arises from (elastic) collisions of particles. A particle hitting a wall head on with momentum $p = mv$ results in a transfer of momentum to the wall of $2mv$. Using this concept, and assuming isotropy for the particle momenta, it can be shown that

$$\boxed{P = \zeta n \langle E \rangle}$$

where $\zeta = 2/3$ ($\zeta = 1/3$) in the case of a non-relativistic (relativistic) fluid, and

$$\langle E \rangle = \int_0^\infty E \mathcal{P}(E) dE$$

is the average, translational energy of the particles. In the case of our ideal (non-relativistic) fluid,

$$\langle E \rangle = \left\langle \frac{p^2}{2m} \right\rangle = \int_0^\infty \frac{p^2}{2m} \mathcal{P}(p) dp = \frac{3}{2} k_B T$$

Hence, we find that the **EoS for an ideal gas** is indeed given by

$$P = \frac{2}{3} n \langle E \rangle = n k_B T = \frac{k_B T}{\mu m_p} \rho$$

Specific Internal Energy: the internal energy per unit mass for an ideal gas is

$$\varepsilon = \frac{\langle E \rangle}{\mu m_p} = \frac{3}{2} \frac{k_B T}{\mu m_p}$$

Actually, the above derivation is only valid for a true ‘ideal gas’, in which the particles are point particles. More generally,

$$\varepsilon = \frac{1}{\gamma - 1} \frac{k_B T}{\mu m_p}$$

where γ is the **adiabatic index**, which for an ideal gas is equal to $\gamma = (q+5)/(q+3)$, with q the internal degrees of freedom of the fluid particles: $q = 0$ for point particles (resulting in $\gamma = 5/3$), while diatomic particles have $q = 2$ (at sufficiently low temperatures, such that they only have rotational, and no vibrational degrees of freedom). The fact that $q = 2$ in that case arises from the fact that a diatomic molecule only has two relevant rotation axes; the third axis is the symmetry axis of the molecule, along which the molecule has negligible (zero in case of point particles) moment of inertia. Consequently, rotation around this symmetry axis carries no energy.

Photon gas: Having discussed the EoS of an ideal gas, we now focus on a gas of photons. Photons have energy $E = h\nu$ and momentum $p = E/c = h\nu/c$, with h the Planck constant.

Black Body: an idealized physical body that absorbs all incident radiation. A black body (BB) in thermal equilibrium emits electro-magnetic radiation called **black body radiation**.

The **spectral number density distribution** of BB photons is given by

$$n_\gamma(\nu, T) = \frac{8\pi\nu^2}{c^3} \frac{1}{e^{h\nu/k_B T} - 1}$$

which implies a **spectral energy distribution**

$$u(\nu, T) = n_\gamma(\nu, T) h\nu = \frac{8\pi h\nu^3}{c^3} \frac{1}{e^{h\nu/k_B T} - 1}$$

and thus an **energy density** of

$$u(T) = \int_0^\infty u(\nu, T) d\nu = \frac{4\sigma_{\text{SB}}}{c} T^4 \equiv a_r T^4$$

where

$$\sigma_{\text{SB}} = \frac{2\pi^5 k_B^4}{15h^3 c^2}$$

is the **Stefan-Boltzmann constant** and $a_r \simeq 7.6 \times 10^{-15} \text{erg cm}^{-3} \text{K}^{-4}$ is called the **radiation constant**.

Radiation Pressure: when the photons are reflected off a wall, or when they are absorbed and subsequently re-emitted by that wall, they transfer twice their momentum in the normal direction to that wall. Since photons are relativistic, we have that **the EoS for a photon gas** is given by

$$\boxed{P = \frac{1}{3} n \langle E \rangle = \frac{1}{3} n_\gamma \langle h\nu \rangle = \frac{1}{3} u(T) = \frac{aT^4}{3}}$$

where we have used that $u(T) = n_\gamma \langle E \rangle$.

Quantum Statistics: according to quantum statistics, a collection of many indistinguishable elementary particles in **thermal equilibrium** has a momentum distribution given by

$$f(\vec{p}) d^3\vec{p} = \frac{g}{h^3} \left[\exp \left(\frac{E(p) - \mu}{k_B T} \right) \pm 1 \right]^{-1} d^3\vec{p}$$

where the signature $\pm$ takes the positive sign for fermions (which have half-integer spin), in which case the distribution is called the **Fermi-Dirac distribution**, and the negative sign for bosons (particles with zero or integer spin), in which case the distribution is called the **Bose-Einstein distribution**. The factor g is the **spin degeneracy factor**, which expresses the number of spin states the particles can have ($g = 1$ for neutrinos, $g = 2$ for photons and charged leptons, and $g = 6$

for quarks). Finally, μ is called the **chemical potential**, and is a form of potential energy that is related (in a complicated way) to the number density and temperature of the particles (see Appendix G).

Classical limit: In the limit where the mean interparticle separation is much larger than the de Broglie wavelength of the particles, so that quantum effects (e.g., Heisenberg's uncertainty principle) can be ignored, the above distribution function of momenta can be accurately approximated by the **Maxwell-Boltzmann distribution**.

Heisenberg's Uncertainty Principle: $\Delta x \Delta p_x > h$ (where $h = 6.63 \times 10^{-27} \text{ g cm}^2 \text{ s}^{-1}$ is Planck's constant). One interpretation of this quantum principle is that phase-space is quantized; no particle can be localized in a phase-space element smaller than the fundamental element

$$\Delta x \Delta y \Delta z \Delta p_x \Delta p_y \Delta p_z = h^3$$

Pauli Exclusion Principle: no more than one fermion of a given spin state can occupy a given phase-space element h^3 . Hence, for electrons, which have $g = 2$, the maximum phase-space density is $2/h^3$.

Degeneracy: When compressing and/or cooling a fermionic gas, at some point all possible low momentum states are occupied. Any further compression therefore results in particles occupying high (but the lowest available) momentum states. Since particle momentum is ultimately responsible for pressure, this degeneracy manifests itself as an extremely high pressure, known as **degeneracy pressure**.

Fermi Momentum: Consider a **fully degenerate** gas of electrons of electron density n_e . It will have fully occupied the part of phase-space with momenta $p \leq p_F$. Here p_F is the maximum momentum of the particles, and is called the **Fermi momentum**. The energy corresponding to the Fermi momentum is called the **Fermi energy**, E_F and is equal to $p_F^2/2m$ in the case of a non-relativistic gas, and $p_F c$ in the case of a relativistic gas.

Let V_x be the volume occupied in configuration space, and $V_p = \frac{4}{3}\pi p_F^3$ the volume occupied in momentum space. If the total number of particles is N , and the gas is

fully degenerate, then

$$V_x V_p = \frac{N}{2} h^3$$

Using that $n_e = N/V_x$, we find that

$$p_F = \left(\frac{3}{8\pi} n_e \right)^{1/3} h$$

EoS of Non-Relativistic, Degenerate Gas: Using the information above, it is relatively straightforward (see Problem Sets) to compute the EoS for a fully degenerate gas. Using that for a non-relativistic fluid $E = p^2/2m$ and $P = \frac{2}{3} n \langle E \rangle$, while degeneracy implies that

$$\langle E \rangle = \frac{1}{N} \int_0^{E_F} E N(E) dE = \frac{1}{N} \int_0^{p_F} \frac{p^2}{2m} \frac{2}{h^3} V_x 4\pi p^2 dp = \frac{3}{5} \frac{p_F^2}{2m}$$

we obtain that

$$P = \frac{1}{20} \left(\frac{3}{\pi} \right)^{2/3} \frac{h^2}{m^{8/3}} \rho^{5/3}$$

EoS of Relativistic, Degenerate Gas: In the case of a relativistic, degenerate gas, we use the same procedure as above. However, this time we have that $P = \frac{1}{3} n \langle E \rangle$ while $E = pc$, which results in

$$P = \frac{1}{8} \left(\frac{3}{\pi} \right)^{1/3} \frac{c h}{m^{4/3}} \rho^{4/3}$$

White Dwarfs and the Chandrasekhar limit: White dwarfs are the end-states of stars with mass low enough that they don't form a neutron star. When the pressure support from nuclear fusion in a star comes to a halt, the core will start to contract until degeneracy pressure kicks in. The star consists of a fully ionized plasma. Assume for simplicity that the plasma consists purely of hydrogen, so that the number density of protons is equal to that of electrons: $n_p = n_e$. Because of **equipartition**

$$\frac{p_p^2}{2m_p} = \frac{p_e^2}{2m_e}$$

Since $m_p \gg m_e$ we have also that $p_p \gg p_e$ (in fact $p_p/p_e = \sqrt{m_p/m_e} \simeq 43$). Consequently, when cooling or compressing the core of a star, the electrons will become degenerate well before the protons do. Hence, white dwarfs are held up against collapse by the **degeneracy pressure from electrons**. Since the electrons are typically non-relativistic, the EoS of the white dwarf is: $P \propto \rho^{5/3}$. If the white dwarf becomes more and more massive (i.e., because it is accreting mass from a companion star), the Pauli-exclusion principle causes the Fermi momentum, p_F , to increase to relativistic values. This **softens** the EoS towards $P \propto \rho^{4/3}$. Such an equation of state is too soft to stabilize the white dwarf against gravitational collapse; the white dwarf collapses until it becomes a **neutron star**, at which stage it is supported against further collapse by the degeneracy pressure from neutrons. This happens when the mass of the white dwarf reaches $M_{\text{lim}} \simeq 1.44M_\odot$, the so-called **Chandrasekhar limit**.

	Non-Relativistic	Relativistic
non-degenerate	$P \propto \rho T$	$P \propto T^4$
degenerate	$P \propto \rho^{5/3}$	$P \propto \rho^{4/3}$

Summary of equations of state for different kind of fluids

CHAPTER 7

Vorticity & Circulation

Vorticity: The vorticity of a flow is defined as the curl of the velocity field:

$$\text{vorticity : } \vec{w} = \nabla \times \vec{u}$$

It is a **microscopic** measure of rotation (vector) at a given point in the fluid, which can be envisioned by placing a paddle wheel into the flow. If it spins about its axis at a rate Ω , then $w = |\vec{w}| = 2\Omega$.

Circulation: The circulation around a closed contour C is defined as the line integral of the velocity along that contour:

$$\text{circulation : } \Gamma_C = \oint_C \vec{u} \cdot d\vec{l} = \int_S \vec{w} \cdot d\vec{S}$$

where S is an *arbitrary* surface bounded by C . The circulation is a **macroscopic** measure of rotation (scalar) for a finite area of the fluid.

Irrotational fluid: An irrotational fluid is defined as being curl-free; hence, $\vec{w} = 0$ and therefore $\Gamma_C = 0$ for any C .

Vortex line: a line that points in the direction of the vorticity vector. Hence, a vortex line relates to $\vec{w}$, as a streamline relates to $\vec{u}$ (cf. Chapter 2).

Vortex tube: a bundle of vortex lines. The circularity of a curve C is proportional to the number of vortex lines that thread the enclosed area.

In an **inviscid** fluid the vortex lines/tubes move **with** the fluid: a vortex line anchored to some fluid element remains anchored to that fluid element.

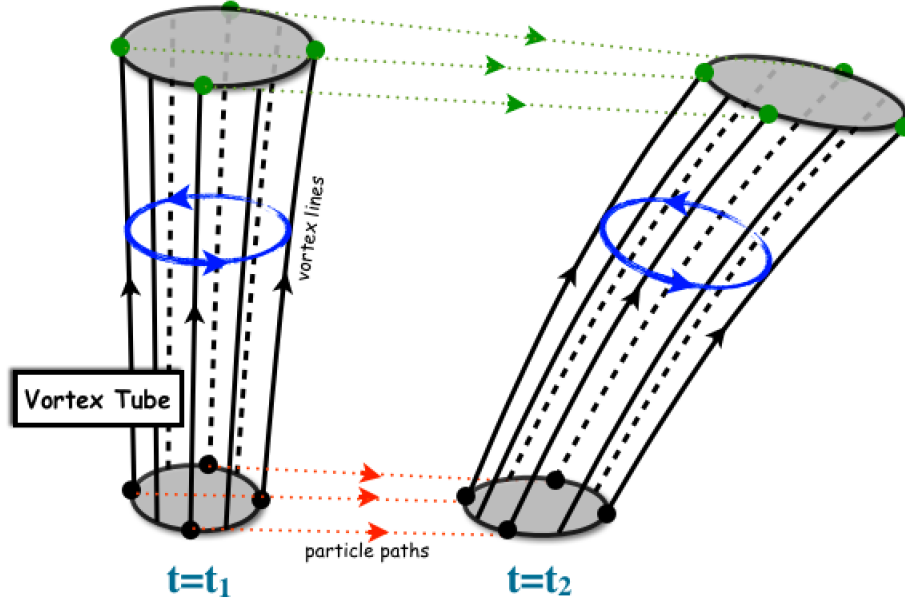


Figure 4: *Evolution of a vortex tube. Solid dots correspond to fluid elements. Due to the shear in the velocity field, the vortex tube is stretched and tilted. However, as long as the fluid is inviscid and barotropic Kelvin's circularity theorem assures that the circularity is conserved with time. In addition, since vorticity is divergence-free ('solenoidal'), the circularity along different cross sections of the same vortex-tube is the same.*

Vorticity equation: The Navier-Stokes momentum equations, in the absence of bulk viscosity, in Eulerian vector form, are given by

$$\frac{\partial \vec{u}}{\partial t} + (\vec{u} \cdot \nabla) \vec{u} = -\frac{\nabla P}{\rho} - \nabla \Phi + \nu \left[\nabla^2 \vec{u} + \frac{1}{3} \nabla(\nabla \cdot \vec{u}) \right]$$

Using the vector identity $(\vec{u} \cdot \nabla) \vec{u} = \frac{1}{2} \nabla u^2 + (\nabla \times \vec{u}) \times \vec{u} = \nabla(u^2/2) - \vec{u} \times \vec{w}$ allows us to rewrite this as

$$\frac{\partial \vec{u}}{\partial t} - \vec{u} \times \vec{w} = -\frac{\nabla P}{\rho} - \nabla \Phi - \frac{1}{2} \nabla u^2 + \nu \left[\nabla^2 \vec{u} + \frac{1}{3} \nabla(\nabla \cdot \vec{u}) \right]$$

If we now take the curl on both sides of this equation, and we use that $\text{curl}(\text{grad } S) = 0$ for any scalar field S , and that $\nabla \times (\nabla^2 \vec{A}) = \nabla^2(\nabla \times \vec{A})$, we obtain the **vorticity equation**:

$$\boxed{\frac{\partial \vec{w}}{\partial t} = \nabla \times (\vec{u} \times \vec{w}) - \nabla \times \left(\frac{\nabla P}{\rho} \right) + \nu \nabla^2 \vec{w}}$$

To write this in Lagrangian form, we first use that $\nabla \times (S \vec{A}) = \nabla S \times \vec{A} + S (\nabla \times \vec{A})$ [see Appendix A] to write

$$\nabla \times \left(\frac{1}{\rho} \nabla P \right) = \nabla \left(\frac{1}{\rho} \right) \times \nabla P + \frac{1}{\rho} (\nabla \times \nabla P) = \frac{\rho \nabla(1) - 1 \nabla \rho}{\rho^2} \times \nabla P = \frac{\nabla P \times \nabla \rho}{\rho^2}$$

where we have used, once more, that $\text{curl}(\text{grad } S) = 0$. Next, using the vector identities from Appendix A, we write

$$\nabla \times (\vec{w} \times \vec{u}) = \vec{w}(\nabla \cdot \vec{u}) - (\vec{w} \cdot \nabla) \vec{u} - \vec{u}(\nabla \cdot \vec{w}) + (\vec{u} \cdot \nabla) \vec{w}$$

The third term vanishes because $\nabla \cdot \vec{w} = \nabla \cdot (\nabla \times \vec{u}) = 0$. Hence, using that $\partial \vec{w} / \partial t + (\vec{u} \cdot \nabla) \vec{w} = d\vec{w}/dt$ we finally can write the **vorticity equation in Lagrangian form**:

$$\boxed{\frac{d\vec{w}}{dt} = (\vec{w} \cdot \nabla) \vec{u} - \vec{w}(\nabla \cdot \vec{u}) + \frac{\nabla \rho \times \nabla P}{\rho^2} + \nu \nabla^2 \vec{w}}$$

This equation describes how the vorticity of a fluid element evolves with time. We now describe the various terms of the *rhs* of this equation in turn:

- $(\vec{w} \cdot \nabla) \vec{u}$: This term represents the **stretching** and **tilting** of vortex tubes due to velocity gradients. To see this, we pick $\vec{w}$ to be pointing in the z -direction. Then

$$(\vec{w} \cdot \nabla) \vec{u} = w_z \frac{\partial \vec{u}}{\partial z} = w_z \frac{\partial u_x}{\partial z} \vec{e}_x + w_z \frac{\partial u_y}{\partial z} \vec{e}_y + w_z \frac{\partial u_z}{\partial z} \vec{e}_z +$$

The first two terms on the *rhs* describe the tilting of the vortex tube, while the third term describes the stretching.

- $\vec{w}(\nabla \cdot \vec{u})$: This term describes **stretching** of vortex tubes due to flow **compressibility**. This term is zero for an incompressible fluid or flow ($\nabla \cdot \vec{u} = 0$). Note that, again under the assumption that the vorticity is pointing in the z -direction,

$$\vec{w}(\nabla \cdot \vec{u}) = w_z \left[\frac{\partial u_x}{\partial x} + \frac{\partial u_y}{\partial y} + \frac{\partial u_z}{\partial z} \right] \vec{e}_z$$

- $(\nabla \rho \times \nabla P)/\rho^2$: This is the **baroclinic** term. It describes the production of vorticity due to a misalignment between pressure and density gradients. This term is zero for a **barotropic** EoS: if $P = P(\rho)$ the pressure and density gradients are parallel so that $\nabla P \times \nabla \rho = 0$. Obviously, this baroclinic term also vanishes for an incompressible fluid ($\nabla \rho = 0$) or for an isobaric fluid ($\nabla P = 0$). The baroclinic term is responsible, for example, for creating vorticity in pyroclastic flows (see Fig. 5).
- $\nu \nabla^2 \vec{w}$: This term describes the **diffusion** of vorticity due to **viscosity**, and is obviously zero for an inviscid fluid ($\nu = 0$). Typically, viscosity generates/creates vorticity at a bounding surface: due to the *no-slip* boundary condition shear arises giving rise to vorticity, which is subsequently diffused into the fluid by the viscosity. In the interior of a fluid, no new vorticity is generated; rather, viscosity diffuses and dissipates vorticity.
- $\nabla \times \vec{F}$: There is a fifth term that can create vorticity, which however does not appear in the vorticity equation above. The reason is that we assumed that the only external force is gravity, which is a conservative force and can therefore be written as the gradient of a (gravitational) potential. More generally, though, there may be non-conservative, external body forces present, which would give rise to a $\nabla \times \vec{F}$ term in the rhs of the vorticity equation. An example of a non-conservative force creating vorticity is the **Coriolis force**, which is responsible for creating hurricanes.

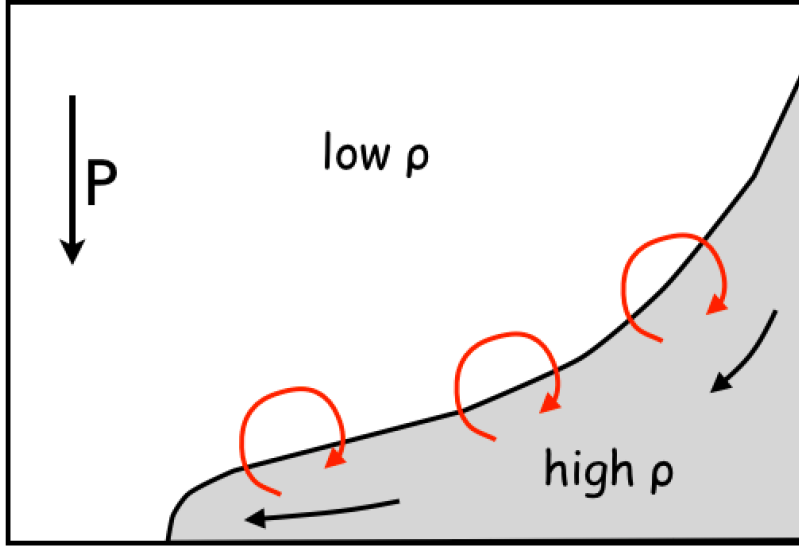


Figure 5: *The baroclinic creation of vorticity in a pyroclastic flow. High density fluid flows down a mountain and shoves itself under lower-density material, thus creating non-zero baroclinicity.*

Using the definition of **circulation**, it can be shown (here without proof) that

$$\frac{d\Gamma}{dt} = \int_S \left[\frac{\partial \vec{w}}{\partial t} + \nabla \times (\vec{w} \times \vec{u}) \right] \cdot d\vec{S}$$

Using the **vorticity equation**, this can be rewritten as

$$\boxed{\frac{d\Gamma}{dt} = \int_S \left[\frac{\nabla \rho \times \nabla P}{\rho^2} + \nu \nabla^2 \vec{w} + \nabla \times \vec{F} \right] \cdot d\vec{S}}$$

where, for completeness, we have added in the contribution of an external force $\vec{F}$ (which vanishes if $\vec{F}$ is conservative). Using Stokes' Curl Theorem (see Appendix B) we can also write this equation in a line-integral form as

$$\boxed{\frac{d\Gamma}{dt} = - \oint \frac{\nabla P}{\rho} \cdot d\vec{l} + \nu \oint \nabla^2 \vec{u} \cdot d\vec{l} + \oint \vec{F} \cdot d\vec{l}}$$

which is the form that is more often used.

NOTE: By comparing the equations expressing $d\vec{w}/dt$ and $d\Gamma/dt$ it is clear that the stretching and tilting terms present in the equation describing $d\vec{w}/dt$, are absent in the equation describing $d\Gamma/dt$. This implies that stretching and tilting changes the vorticity, but keeps the circularity invariant. This is basically the first theorem of Helmholtz described below.

Kelvin's Circulation Theorem: The number of vortex lines that thread any element of area that moves with the fluid (i.e., the circulation) remains unchanged in time for an inviscid, barotropic fluid, in the absence of non-conservative forces.

The proof of **Kelvin's Circulation Theorem** is immediately evident from the above equation, which shows that $d\Gamma/dt = 0$ if the fluid is both **inviscid** ($\nu = 0$), **barotropic** ($P = P(\rho) \Rightarrow \nabla\rho \times \nabla P = 0$), and there are no non-conservative forces ($\vec{F} = 0$).

We end this chapter on vorticity and circulation with the three theorems of Helmholtz, which hold in the absence of non-conservative forces (i.e., $\vec{F} = 0$).

Helmholtz Theorem 1: The **strength** of a vortex tube, which is defined as the circularity of the circumference of any cross section of the tube, is constant along its length. This theorem holds for **any** fluid, and simply derives from the fact that the vorticity field is **divergence-free** (we say **solenoidal**): $\nabla \cdot \vec{w} = \nabla \cdot (\nabla \times \vec{u}) = 0$. To see this, use Gauss' divergence theorem to write that

$$\int_V \nabla \cdot \vec{w} dV = \int_S \vec{w} \cdot d^2S = 0$$

Here V is the volume of a subsection of the vortex tube, and S is its bounding surface. Since the vorticity is, by definition, perpendicular to S along the sides of the tube, the only non-vanishing components to the surface integral come from the areas at the top and bottom of the vortex tube; i.e.

$$\int_S \vec{w} \cdot d^2\vec{S} = \int_{A_1} \vec{w} \cdot (-\hat{n}) dA + \int_{A_2} \vec{w} \cdot \hat{n} dA = 0$$

where A_1 and A_2 are the areas of the cross sections that bound the volume V of the vortex tube. Using Stokes' curl theorem, we have that

$$\int_A \vec{w} \cdot \hat{n} \, dA = \oint_C \vec{u} \cdot d\vec{l}$$

Hence we have that $\Gamma_{C_1} = \Gamma_{C_2}$ where C_1 and C_2 are the curves bounding A_1 and A_2 , respectively.

Helmholtz Theorem 2: A vortex line cannot end in a fluid. Vortex lines and tubes must appear as closed loops, extend to infinity, or start/end at solid boundaries.

Helmholtz Theorem 3: A barotropic, inviscid fluid that is initially irrotational will remain irrotational in the absence of rotational (i.e., non-conservative) external forces. Hence, such a fluid does not and cannot create vorticity (except across curved shocks, see Chapter 10).

The proof of Helmholtz' third theorem is straightforward. According to Kelvin's circulation theorem, a barotropic, inviscid fluid has $d\Gamma/dt = 0$ everywhere. Hence,

$$\frac{d\Gamma}{dt} = \int_S \left[\frac{\partial \vec{w}}{\partial t} + \nabla \times (\vec{w} \times \vec{u}) \right] \cdot d^2\vec{S} = 0$$

Since this has to hold for any S , we have that $\partial \vec{w} / \partial t = \nabla \times (\vec{u} \times \vec{w})$. Hence, if $\vec{w} = 0$ initially, the vorticity remains zero for ever.

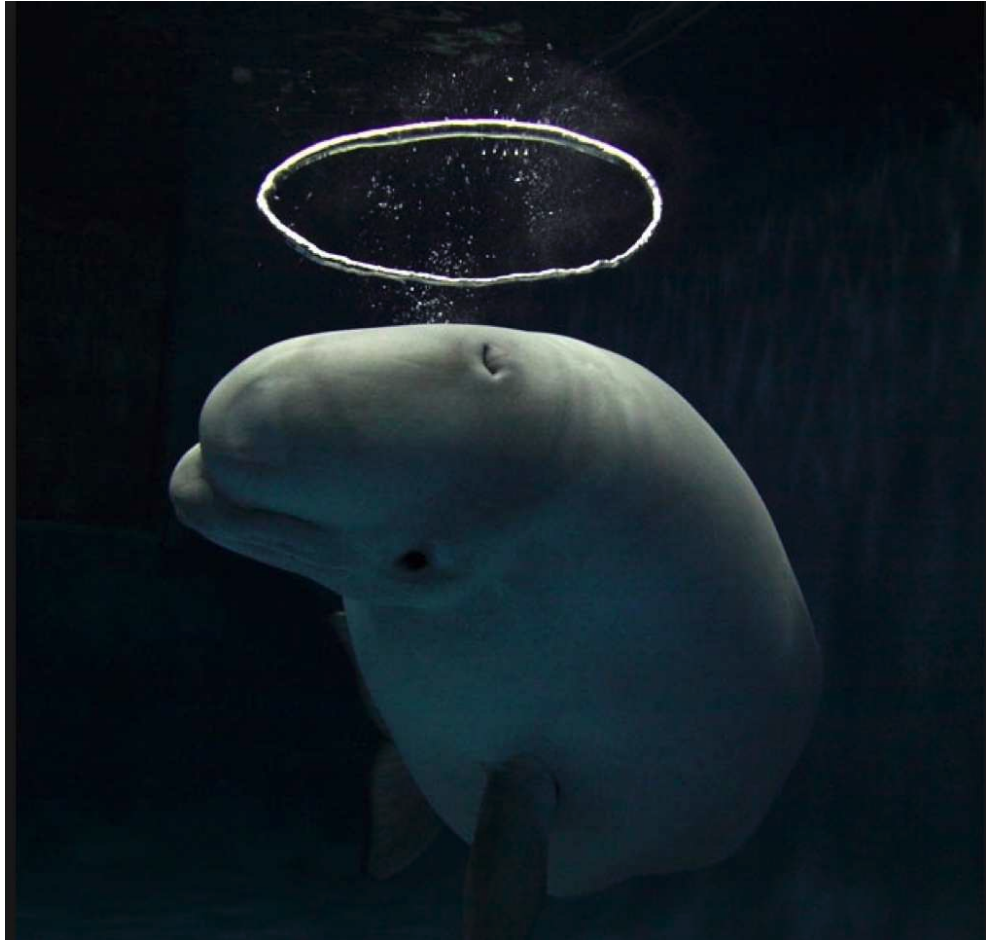


Figure 6: *A beluga whale demonstrating Kelvin's circulation theorem and Helmholtz' second theorem by producing a closed vortex tube under water, made out of air.*

CHAPTER 8

Hydrostatics and Steady Flows

Having derived all the relevant equations for hydrodynamics, we now start examining several specific flows. Since a fully general solution of the Navier-Stokes equation is (still) lacking (this is one of the seven Millenium Prize Problems, a solution of which will earn you \$1,000,000), we can only make progress if we make several assumptions.

We start with arguably the simplest possible flow, namely ‘no flow’. This is the area of **hydrostatics** in which $\vec{u}(\vec{x}, t) = 0$. And since we seek a static solution, we also must have that all $\partial/\partial t$ -terms vanish. Finally, in what follows we shall also ignore radiative processes (i.e., we set $\mathcal{L} = 0$).

Applying these restrictions to the continuity, momentum and energy equations (see box at the end of Chapter 5) yields the following two non-trivial equations:

$$\nabla P = -\rho \nabla \Phi$$

$$\nabla \cdot \vec{F}_{\text{cond}} = 0$$

The first equation is the well known equation of **hydrostatic equilibrium**, stating that the gravitational force is balanced by pressure gradients, while the second equation states that in a static fluid the **conductive flux** needs to be divergence-free.

To further simplify matters, let’s assume (i) spherical symmetry, and (ii) a barotropic equation of state, i.e., $P = P(\rho)$.

The equation of hydrostatic equilibrium now reduces to

$$\frac{dP}{dr} = -\frac{G M(r) \rho(r)}{r^2}$$

In addition, if the gas is self-gravitating (such as in a star) then we also have that

$$\boxed{\frac{dM}{dr} = 4\pi\rho(r)r^2}$$

For a barotropic EoS this is a closed set of equations, and the density profile can be solved for (given proper boundary conditions). Of particular interest in astrophysics, is the case of a **polytropic EoS**: $P \propto \rho^\Gamma$, where Γ is the polytropic index. Note that $\Gamma = 1$ and $\Gamma = \gamma$ for isothermal and adiabatic equations of state, respectively. A spherically symmetric, polytropic fluid in HE is called a polytropic sphere.

Lane-Emden equation: Upon substituting the **polytropic EoS** in the equation of **hydrostatic equilibrium** and using the **Poisson equation**, one obtains a single differential equation that completely describes the structure of the polytropic sphere, known as the **Lane-Emden equation**:

$$\frac{1}{\xi^2} \frac{d}{d\xi} \left(\xi^2 \frac{d\theta}{d\xi} \right) = -\theta^n$$

Here $n = 1/(\Gamma - 1)$ is related to the polytropic index (in fact, confusingly, some texts refer to n as the polytropic index),

$$\xi = \left(\frac{4\pi G \rho_c}{\Phi_0 - \Phi_c} \right)^{1/2} r$$

is a dimensionless radius,

$$\theta = \left(\frac{\Phi_0 - \Phi(r)}{\Phi_0 - \Phi_c} \right)$$

with Φ_c and Φ_0 the values of the gravitational potential at the center ($r = 0$) and at the surface of the star (where $\rho = 0$), respectively. The density is related to θ according to $\rho = \rho_c \theta^n$ with ρ_c the central density.

Solutions to the Lane-Emden equation are called **polytropes of index n** . In general, the Lane-Emden equation has to be solved numerically subject to the boundary conditions $\theta = 1$ and $d\theta/d\xi = 0$ at $\xi = 0$. Analytical solutions exist, however, for $n = 0, 1$, and 5 . Examples of polytropes are stars that are supported by **degeneracy pressure**. For example, a non-relativistic, degenerate equation of state has $P \propto \rho^{5/3}$

(see Chapter 6) and is therefore described by a polytrope of index $n = 3/2$. In the relativistic case $P \propto \rho^{4/3}$ which results in a polytrope of index $n = 3$.

Another polytrope that is often encountered in astrophysics is the **isothermal sphere**, which has $P \propto \rho$ and thus $n = \infty$. It has $\rho \propto r^{-2}$ at large radii, which implies an infinite total mass. If one truncates the isothermal sphere at some radius and embeds it in a medium with external pressure (to prevent the sphere from expanding), it is called a **Bonnor-Ebert sphere**, which is a structure that is frequently used to describe molecular clouds.

Stellar Structure: stars are gaseous spheres in hydrostatic equilibrium (except for radial pulsations, which may be considered perturbations away from HE). The structure of stars is therefore largely governed by the above equation.

However, in general the **equation of state** is of the form $P = P(\rho, T, \{X_i\})$, where $\{X_i\}$ is the set of the abundances of all elements i . The temperature structure of a star and its abundance ratios are governed by **nuclear physics** (which provides the source of energy) and the various **heat transport mechanisms**.

Heat transport in stars: Typically, ignoring abundance gradients, stars have the equation of state of an ideal gas, $P = P(\rho, T)$. This implies that the equations of stellar structure need to be complemented by an equation of the form

$$\boxed{\frac{dT}{dr} = F(r)}$$

Since T is a measure of the internal energy, the rhs of this equation describes the **heat flux**, $F(r)$.

The main heat transport mechanisms in a star are:

- conduction
- convection
- radiation

Note that the fourth heat transport mechanism, advection, is not present in the case of hydrostatic equilibrium, because $\vec{u} = 0$.

Recall from Chapter 4 that the **thermal conductivity** $\mathcal{K} \propto (k_B T)^{1/2}/\sigma$ where σ is the collisional cross section. Using that $k_B T \propto v^2$ and that the mean-free path of the particles is $\lambda_{\text{mfp}} = 1/(n\sigma)$, we have that

$$\mathcal{K} \propto n \lambda_{\text{mfp}} v$$

with v the thermal, microscopic velocity of the particles (recall that $\vec{u} = 0$). Since radiative heat transport in a star is basically the conduction of photons, and since $c \gg v_e$ and the mean-free part of photons is much larger than that of electrons (after all, the cross section for Thomson scattering, σ_T , is much smaller than the typical cross section for Coulomb interactions), we have that in stars *radiation is a far more efficient heat transport mechanism than conduction*. An exception are relativistic, degenerate cores, for which $v_e \sim c$ and photons and electrons have comparable mean-free paths.

Convection: convection only occurs if the **Schwarzschild Stability Criterion** is violated, which happens when the temperature gradient dT/dr becomes too large (i.e., larger than the temperature gradient that would exist if the star was adiabatic; see Chapter 15). If that is the case, convection always dominates over radiation as the most efficient heat transport mechanism. In general, as a rule of thumb, more massive stars are *more radiative* and *less convective*.

Trivia: On average it takes $\sim 200,000$ years for a photon created at the core of the Sun in nuclear burning to make its way to the Sun's photosphere; from there it only takes ~ 8 minutes to travel to the Earth.

Hydrostatic Mass Estimates: Now let us consider the case of an ideal gas, for which

$$P = \frac{k_B T}{\mu m_p} \rho,$$

but this time the gas is not self-gravitating; rather, the gravitational potential may be considered 'external'. A good example is the ICM; the hot gas that permeates clusters. From the EoS we have that

$$\begin{aligned}
\frac{dP}{dr} &= \frac{\partial P}{\partial \rho} \frac{d\rho}{dr} + \frac{\partial P}{\partial T} \frac{dT}{dr} = \frac{P}{\rho} \frac{d\rho}{dr} + \frac{P}{T} \frac{dT}{dr} \\
&= \frac{P}{r} \left[\frac{r}{\rho} \frac{d\rho}{dr} + \frac{r}{T} \frac{dT}{dr} \right] = \frac{P}{r} \left[\frac{d \ln \rho}{d \ln r} + \frac{d \ln T}{d \ln r} \right]
\end{aligned}$$

Substitution of this equation in the equation for Hydrostatic equilibrium (HE) yields

$$\boxed{M(r) = -\frac{k_B T(r) r}{\mu m_p G} \left[\frac{d \ln \rho}{d \ln r} + \frac{d \ln T}{d \ln r} \right]}$$

This equation is often used to measure the ‘hydrostatic’ mass of a galaxy cluster; X-ray measurements can be used to infer $\rho(r)$ and $T(r)$ (after deprojection, which is analytical in the case of spherical symmetry). Substitution of these two radial dependencies in the above equation then yields an estimate for the cluster’s mass profile, $M(r)$. Note, though, that this mass estimate is based on three crucial assumptions: (i) sphericity, (ii) hydrostatic equilibrium, and (iii) an ideal-gas EoS. Clusters typically are not spherical, often are turbulent (such that $\vec{u} \neq 0$, violating the assumption of HE), and can have significant contributions from non-thermal pressure due to magnetic fields, cosmic rays and/or turbulence. Including these non-thermal pressure sources the above equation becomes

$$M(r) = -\frac{k_B T(r) r}{\mu m_p G} \left[\frac{d \ln \rho}{d \ln r} + \frac{d \ln T}{d \ln r} + \frac{P_{\text{nt}}}{P_{\text{th}}} \frac{d \ln P_{\text{nt}}}{d \ln r} \right]$$

were P_{nt} and P_{th} are the non-thermal and thermal contributions to the total gas pressure. Unfortunately, it is extremely difficult to measure P_{nt} reliably, which is therefore often ignored. This may result in systematic biases of the inferred cluster mass (typically called the ‘hydrostatic mass’).

Solar Corona: As a final example of a hydrostatic problem in astrophysics, consider the problem of constructing a static model for the Solar corona.

The Solar corona is a large, spherical region of hot ($T \sim 10^6\text{K}$) plasma extending well beyond its photosphere. Let’s assume that the heat is somehow (magnetic reconnection?) produced in the lower layers of the corona, and try to infer the density, temperature and pressure profiles under the assumption of hydrostatic equilibrium.

We have the boundary condition of the temperature at the base, which we assume to be $T_0 = 3 \times 10^6 \text{K}$, at a radius of $r = r_0 \sim R_\odot \simeq 6.96 \times 10^{10} \text{cm}$. The mass of the corona is negligible, and we therefore have that

$$\begin{aligned} \frac{dP}{dr} &= -\frac{G M_\odot}{r^2} \frac{\mu m_p}{k_B} \frac{P}{T} \\ \frac{d}{dr} \left(\mathcal{K} r^2 \frac{dT}{dr} \right) &= 0 \end{aligned}$$

where we have used the ideal gas EoS to substitute for ρ . Note that the latter of these equations follows from $\nabla \cdot \vec{F} = 0$, which is the energy equation in HE. As we have seen above $\mathcal{K} \propto n \lambda_{\text{mfp}} T^{1/2}$. In a plasma one furthermore has that $\lambda_{\text{mfp}} \propto n^{-1} T^2$, which implies that $\mathcal{K} \propto T^{5/2}$. Hence, the second equation can be written as

$$r^2 T^{5/2} \frac{dT}{dr} = \text{constant}$$

which implies

$$T = T_0 \left(\frac{r}{r_0} \right)^{-2/7}$$

Note that this equation satisfies our boundary condition, and that $T_\infty = \lim_{r \rightarrow \infty} T(r) = 0$. Substituting this expression for T in the HE equation yields

$$\frac{dP}{P} = -\frac{G M_\odot \mu m_p}{k_B T_0 r_0^{2/7}} \frac{dr}{r^{12/7}}$$

Solving this ODE under the boundary condition that $P = P_0$ at $r = r_0$ yields

$$P = P_0 \exp \left[\frac{7}{5} \frac{G M_\odot \mu m_p}{k_B T_0 r_0} \left\{ \left(\frac{r}{r_0} \right)^{-5/7} - 1 \right\} \right]$$

Note that

$$\lim_{r \rightarrow \infty} P = P_0 \exp \left[-\frac{7}{5} \frac{G M_\odot \mu m_p}{k_B T_0 r_0} \right] \neq 0$$

Hence, you need an external pressure to confine the corona. Well, that seems OK, given that the Sun is embedded in an ISM, whose pressure we can compute taking characteristic values for the warm phase ($T \sim 10^4\text{K}$ and $n \sim 1\text{ cm}^{-3}$). Note that the other phases (cold and hot) have the same pressure. Plugging in the numbers, we find that

$$\frac{P_\infty}{P_{\text{ISM}}} \sim 10 \frac{\rho_0}{\rho_{\text{ISM}}}$$

Since $\rho_0 \gg \rho_{\text{ISM}}$ we thus infer that the ISM pressure falls short, by orders of magnitude, to be able to confine the corona....

As first inferred by Parker in 1958, the correct implication of this puzzling result is that a hydrostatic corona is impossible; instead, Parker made the daring suggestion that there should be a **solar wind**, which was observationally confirmed a few years later.

Having addressed hydrostatics ('no flow'), we now consider the next simplest flow; **steady flow**, which is characterised by $\vec{u}(\vec{x}, t) = \vec{u}(\vec{x})$. For steady flow $\partial\vec{u}/\partial t = 0$, and fluid elements move along the **streamlines** (see Chapter 2).

Using the vector identity $(\vec{u} \cdot \nabla) \vec{u} = \frac{1}{2} \nabla u^2 + (\nabla \times \vec{u}) \times \vec{u} = \nabla(u^2/2) - \vec{u} \times \vec{w}$, allows us to write the Navier-Stokes equation for a steady flow of ideal fluid as

$$\boxed{\nabla \left(\frac{u^2}{2} + \Phi \right) + \frac{\nabla P}{\rho} - \vec{u} \times \vec{w} = 0}$$

This equation is known as **Crocco's theorem**. In order to write this in a more 'useful' form, we first proceed to demonstrate that $\nabla P/\rho$ can be written in terms of the gradients of the **specific enthalpy**, h , and the **specific entropy**, s :

The **enthalpy**, H , is a measure for the total energy of a thermodynamic system that includes the **internal** energy, U , and the amount of energy required to make room for it by displacing its environment and establishing its volume and pressure:

$$H = U + PV$$

The differential of the enthalpy can be written as

$$dH = dU + P dV + V dP$$

Using the first law of thermodynamics, according to which $dU = dQ - P dV$, and the second law of thermodynamics, according to which $dQ = T dS$, we can rewrite this as

$$dH = T dS + V dP$$

which, in specific form, becomes

$$dh = T ds + \frac{dP}{\rho}$$

(i.e., we have $s = S/m$). This relation is one of the **Gibbs relations** frequently encountered in thermodynamics. NOTE: for completeness, we point out that this expression ignores changes in the chemical potential (see Appendix J).

The above expression for dh implies that

$$\boxed{\frac{\nabla P}{\rho} = \nabla h - T \nabla s}$$

(for a formal proof, see at the end of this chapter). Now recall from the previous chapter on vorticity that the **baroclinic term** is given by

$$\nabla \times \left(\frac{\nabla P}{\rho} \right) = \frac{\nabla \rho \times \nabla P}{\rho^2}$$

Using the above relation, and using that the curl of the gradient of a scalar vanishes, we can rewrite this baroclinic term as $\nabla \times (T \nabla s)$. This implies that *one can create vorticity by creating a gradient in (specific) entropy!* One way to do this, which is one of the most important mechanisms of creating vorticity in astrophysics, is via **curved shocks**; when an irrotational, isentropic fluid comes across a curved shock, different streamlines will experience a different jump in entropy (Δs will depend on

the angle under which you cross the shock). Hence, in the post-shocked gas there will be a gradient in entropy, and thus vorticity.

Intermezzo: isentropic vs. adiabatic

We consider a flow to be **isentropic** if it conserves (specific) entropy, which implies that $ds/dt = 0$. Note that an ideal fluid is a fluid without dissipation (viscosity) and conduction (heat flow). Hence, any flow of ideal fluid is isentropic. A fluid is said to be isentropic if $\nabla s = 0$. A process is said to be **adiabatic** if $dQ/dt = 0$. Note that, according to the second law of thermodynamics, $TdS \geq dQ$. Equality only holds for a **reversible** process; in other words, only if a process is adiabatic and reversible do we call it isentropic. An irreversible, adiabatic process, therefore, can still create entropy.

Using the momentum equation for a steady, ideal fluid, and substituting $\nabla P/\rho \rightarrow \nabla h - T \nabla s$, we obtain

$$\nabla B = T \nabla s + \vec{u} \times \vec{w}$$

where we have introduced the **Bernoulli function**

$$B \equiv \frac{u^2}{2} + \Phi + h = \frac{u^2}{2} + \Phi + \varepsilon + P/\rho$$

which obviously is a measure of energy. The above equation is sometimes referred to as **Crocco's theorem**. It relates entropy gradients to vorticity and gradients in the Bernoulli function.

Let's investigate what happens to the Bernoulli function for an ideal fluid in a steady flow. Since we are in a steady state we have that

$$\frac{dB}{dt} = \frac{\partial B}{\partial t} + (\vec{u} \cdot \nabla)B = (\vec{u} \cdot \nabla)B$$

Next we use that

$$\begin{aligned}
(\vec{u} \cdot \nabla)B &= (\vec{u} \cdot \nabla) \left[\frac{u^2}{2} + \Phi + h \right] \\
&= (\vec{u} \cdot \nabla) \left[\frac{u^2}{2} + \Phi \right] + \vec{u} \cdot \frac{\nabla P}{\rho} + T (\vec{u} \cdot \nabla s) \\
&= (\vec{u} \cdot \left[\nabla \left(\frac{u^2}{2} + \Phi \right) + \frac{\nabla P}{\rho} \right] + T (\vec{u} \cdot \nabla s) \\
&= \vec{u} \cdot (\vec{u} \times \vec{w}) + T (\vec{u} \cdot \nabla s) \\
&= 0
\end{aligned}$$

Here we have used that the cross-product of $\vec{u}$ and $\vec{w}$ is perpendicular to $\vec{u}$, and that in an ideal fluid $\vec{u} \cdot \nabla s = 0$. The latter follow from the fact that in an ideal fluid $ds/dt = 0$, and the fact that $ds/dt = \partial s/\partial t + \vec{u} \cdot \nabla s$. Since all $\partial/\partial t$ terms vanish for a steady flow, we see that $\vec{u} \cdot \nabla s = 0$ for a steady flow of ideal fluid. And as a consequence, we thus also have that

$$\boxed{\frac{dB}{dt} = 0}$$

Hence, *in a steady flow of ideal fluid, the Bernoulli function is conserved*. Using the definition of the Bernoulli function we can write this as

$$\frac{dB}{dt} = \vec{u} \cdot \frac{d\vec{u}}{dt} + \frac{d\Phi}{dt} + T \frac{ds}{dt} + \frac{1}{\rho} \frac{dP}{dt} = 0$$

Since $ds/dt = 0$ for an ideal fluid, we have that if the flow is such that the gravitational potential along the flow doesn't change significantly (such that $d\Phi/dt \simeq 0$), we find that

$$\boxed{\vec{u} \cdot \frac{d\vec{u}}{dt} = -\frac{1}{\rho} \frac{dP}{dt}}$$

This is known as **Bernoulli's theorem**, and states that as the speed of a steady flow increases, the internal pressure of the ideal fluid must decrease. Applications of Bernoulli's theorem discussed in class include the shower curtain and the pitot tube (a flow measurement device used to measure fluid flow velocity).

Potential flow: The final flow to consider in this chapter is potential flow. Consider an irrotational flow, which satisfies $\vec{w} \equiv \nabla \times \vec{u} = 0$ everywhere. This implies that there is a scalar function, $\phi_u(x)$, such that $\vec{u} = \nabla\phi_u$, which is why $\phi_u(x)$ is called the **velocity potential**. The corresponding flow $\vec{u}(\vec{x})$ is called potential flow.

If the fluid is ideal (i.e., $\nu = \mathcal{K} = 0$), and barotropic or isentropic, such that the flow fluid has vanishing baroclinicity, then **Kelvin’s circulation theorem** assures that the flow will remain irrotational throughout (no vorticity can be created), provided that all forces acting on the fluid are conservative.

If the fluid is **incompressible**, in addition to being irrotational, then we have that both the curl and the divergence of the velocity field vanish. This implies that

$$\nabla \cdot \vec{u} = \nabla^2 \phi_u = 0$$

This is the well known **Laplace equation**, familiar from electrostatics. Mathematically, this equation is of the elliptic PDE type which requires well defined boundary conditions in order for a solution to both exist and be unique. A classical case of potential flow is the flow around a solid body placed in a large fluid volume. In this case, an obvious boundary condition is the one stating that the velocity component perpendicular to the surface of the body at the body (assumed at rest) is zero. This is called a *Neumann* boundary condition and is given by

$$\frac{\partial \phi_u}{\partial n} = \vec{n} \cdot \nabla \phi_u = 0$$

with $\vec{n}$ the normal vector. The Laplace equation with this type of boundary condition constitutes a well-posed problem with a unique solution. An example of potential flow around a solid body is shown in Fig. 2 in Chapter 2. We will not examine any specific examples of potential flow, as this means having to solve a Laplace equation, which is purely a mathematical exercise. We end, though, by pointing out that real fluids are never perfectly inviscid (ideal fluids don’t exist). And any flow past a surface involves a boundary layer inside of which viscosity creates vorticity (due to no-slip boundary condition, which states that the tangential velocity at the surface of the body must vanish). Hence, potential flow can never fully describe the flow around a solid body; otherwise one would run into **d’Alembert’s paradox** which is that steady potential flow around a body exerts zero force on the body; in other words, it costs no energy to move a body through the fluid at constant speed. We know from everyday experience that this is indeed not true. The solution to the

paradox is that viscosity created in the boundary layer, and subsequently dissipated, results in friction.

Although potential flow around an object can thus never be a full description of the flow, in many cases, the boundary layer is very thin, and away from the boundary layer the solutions of potential flow still provide an accurate description of the flow.

As promised in the text, we end this chapter by demonstrating that

$$dh = T ds + \frac{dP}{\rho} \quad \Longleftrightarrow \quad \nabla h = T \nabla s + \frac{\nabla P}{\rho}$$

To see this, use that the natural variables of h are the specific entropy, s , and the pressure P . Hence, $h = h(s, P)$, and we thus have that

$$dh = \frac{\partial h}{\partial s} ds + \frac{\partial h}{\partial P} dP$$

From a comparison with the previous expression for dh , we see that

$$\frac{\partial h}{\partial s} = T, \quad \frac{\partial h}{\partial P} = \frac{1}{\rho}$$

which allows us to derive

$$\begin{aligned} \nabla h &= \frac{\partial h}{\partial x} \vec{e}_x + \frac{\partial h}{\partial y} \vec{e}_y + \frac{\partial h}{\partial z} \vec{e}_z \\ &= \left(\frac{\partial h}{\partial s} \frac{\partial s}{\partial x} + \frac{\partial h}{\partial P} \frac{\partial P}{\partial x} \right) \vec{e}_x + \left(\frac{\partial h}{\partial s} \frac{\partial s}{\partial y} + \frac{\partial h}{\partial P} \frac{\partial P}{\partial y} \right) \vec{e}_y + \left(\frac{\partial h}{\partial s} \frac{\partial s}{\partial z} + \frac{\partial h}{\partial P} \frac{\partial P}{\partial z} \right) \vec{e}_z \\ &= \frac{\partial h}{\partial s} \left(\frac{\partial s}{\partial x} \vec{e}_x + \frac{\partial s}{\partial y} \vec{e}_y + \frac{\partial s}{\partial z} \vec{e}_z \right) + \frac{\partial h}{\partial P} \left(\frac{\partial P}{\partial x} \vec{e}_x + \frac{\partial P}{\partial y} \vec{e}_y + \frac{\partial P}{\partial z} \vec{e}_z \right) \\ &= T \nabla s + \frac{1}{\rho} \nabla P \end{aligned}$$

which completes our proof.

CHAPTER 9

Viscous Flow and Accretion Flow

As we have seen in our discussion on potential flow in the previous chapter, realistic flow past an object always involves a boundary layer in which viscosity results in vorticity. Even if the viscosity of the fluid is small, the no-slip boundary condition typically implies a region where the shear is substantial, and viscosity thus manifests itself.

In this chapter we examine two examples of **viscous flow**. We start with a well-known example from engineering, known as **Poiseuille-Hagen flow** through a pipe. Although not really an example of astrophysical flow, it is a good illustration of how viscosity manifests itself as a consequence of the **no-slip boundary condition**. The second example that we consider is viscous flow in a thin accretion disk. This flow, which was first worked out in detail in a famous paper by Shakura & Sunyaev in 1973, is still used today to describe accretion disks in AGN and around stars.

Pipe Flow: Consider the steady flow of an incompressible viscous fluid through a circular pipe of radius R_{pipe} and length L . Let ρ be the density of the fluid as it flows through the pipe, and let $\nu = \mu/\rho$ be its **kinetic viscosity**. Since the flow is incompressible, we have that fluid density will be ρ throughout. If we pick a Cartesian coordinate system with the z -axis along the symmetry axis of the cylinder, then the velocity field of our flow is given by

$$\vec{u} = u_z(x, y, z) \vec{e}_z$$

In other words, $u_x = u_y = 0$.

Starting from the **continuity equation**

$$\frac{\partial \rho}{\partial t} + \nabla \cdot \rho \vec{u} = 0$$

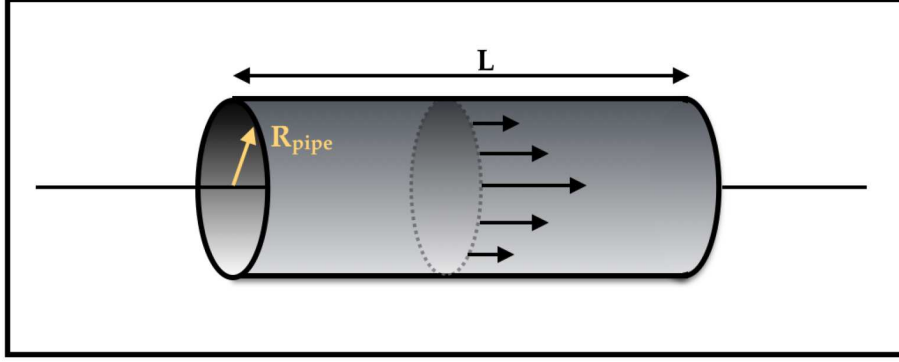


Figure 7: *Poiseuille-Hagen flow of a viscous fluid through a pipe of radius R_{pipe} and length L .*

and using that all partial time-derivatives of a steady flow vanish, we obtain that

$$\frac{\partial \rho u_x}{\partial x} + \frac{\partial \rho u_y}{\partial y} + \frac{\partial \rho u_z}{\partial z} = 0 \quad \Rightarrow \quad \frac{\partial u_z}{\partial z} = 0$$

where we have used that $\partial \rho / \partial z = 0$ because of the incompressibility of the flow. Hence, we can update our velocity field to be $\vec{u} = u_z(x, y) \vec{e}_z$.

Next we write down the **momentum equations** for a steady, incompressible flow, which are given by

$$(\vec{u} \cdot \nabla) \vec{u} = -\frac{\nabla P}{\rho} + \nu \nabla^2 \vec{u} - \nabla \Phi$$

In what follows we assume the pipe to be perpendicular to $\nabla \Phi$, so that we may ignore the last term in the above expression. For the x - and y - components of the momentum equation, one obtains that $\partial P / \partial x = \partial P / \partial y = 0$. For the z -component, we instead have

$$u_z \frac{\partial u_z}{\partial z} = -\frac{1}{\rho} \frac{\partial P}{\partial z} + \nu \nabla^2 u_z$$

Combining this with our result from the continuity equation, we obtain that

$$\boxed{\boxed{\frac{1}{\rho} \frac{\partial P}{\partial z} = \nu \nabla^2 u_z}}$$

Next we use that $\partial P / \partial z$ cannot depend on z ; otherwise u_z would depend on z , but according to the continuity equation $\partial u_z / \partial z = 0$. This means that the pressure

gradient in the z -direction must be constant, which we write as $-\Delta P/L$, where ΔP is the pressure difference between the beginning and end of the pipe, and the minus sign is used to indicate that the fluid pressure declines as it flows through the pipe.

Hence, we have that

$$\nabla^2 u_z = -\frac{\Delta P}{\rho \nu L} = \text{constant}$$

At this point, it is useful to switch to **cylindrical coordinates**, (R, θ, z) , with the z -axis as before. Because of the symmetries involved, we have that $\partial/\partial\theta = 0$, and thus the above expression reduces to

$$\frac{1}{R} \frac{d}{dR} \left(R \frac{du_z}{dR} \right) = -\frac{\Delta P}{\rho \nu L}$$

(see Appendix D). Rewriting this as

$$du_z = -\frac{1}{2} \frac{\Delta P}{\rho \nu L} R dR$$

and integrating from R to R_{pipe} using the no-slip boundary condition that $u_z(R_{\text{pipe}}) = 0$, we finally obtain the flow solution

$$\boxed{u_z(R) = \frac{\Delta P}{4\rho \nu L} [R_{\text{pipe}}^2 - R^2]}$$

This solution is called **Poiseuille flow** or **Poiseuille-Hagen flow**.

As is evident from the above expression, for a given pressure difference ΔP , the flow speed $u \propto \nu^{-1}$ (i.e., a more viscous fluid will flow slower). In addition, for a given fluid viscosity, applying a larger pressure difference ΔP results in a larger flow speed ($u \propto \Delta P$).

Now let us compute the amount of fluid that flows through the pipe per unit time:

$$\dot{M} = 2\pi \int_0^{R_{\text{pipe}}} \rho u_z(R) R dR = \frac{\pi}{8} \frac{\Delta P}{\nu L} R_{\text{pipe}}^4$$

Note the strong dependence on the pipe radius; this makes it clear that a clogging of the pipe has a drastic impact on the mass flow rate (relevant for both arteries and oil-pipelines). The above expression also gives one a relatively easy method to measure

the viscosity of a fluid: take a pipe of known R_{pipe} and L , apply a pressure difference ΔP across the pipe, and measure the mass flow rate, $\dot{M}$; the above expression allows one to then compute ν .

The Poiseuille velocity flow field has been experimentally confirmed, *but only for slow flow!* When $|\vec{u}|$ gets too large (i.e., ΔP is too large), then the flows becomes irregular in time and space; turbulence develops and $|\vec{u}|$ drops due to the enhanced drag from the turbulence. This will be discussed in more detail in Chapter 12.

Accretion Disks: We now move to a viscous flow that is more relevant for astrophysics; accretion flow. Consider a thin accretion disk surrounding an accreting object of mass $M_{\bullet} \gg M_{\text{disk}}$ (such that we may ignore the disk's self-gravity). Because of the symmetries involved, we adopt **cylindrical coordinates**, (R, θ, z) , with the z -axis perpendicular to the disk. We also have that $\partial/\partial\theta$ is zero, and we set $u_z = 0$ throughout.

We expect u_θ to be the main velocity component, with a small u_R component representing the radial accretion flow. We also take the flow to be **incompressible**.

Let's start with the **continuity equation**, which in our case reads

$$\frac{\partial \rho}{\partial t} + \frac{1}{R} \frac{\partial}{\partial R} (R \rho u_R) = 0$$

(see Appendix D for how to express the divergence in cylindrical coordinates).

Next up is the **Navier-Stokes equations**. For now, we only consider the θ -component, which is given by

$$\begin{aligned} \frac{\partial u_\theta}{\partial t} + u_R \frac{\partial u_\theta}{\partial R} + \frac{u_\theta}{R} \frac{\partial u_\theta}{\partial \theta} + u_z \frac{\partial u_\theta}{\partial z} + \frac{u_R u_\theta}{R} = -\frac{1}{\rho} \frac{\partial P}{\partial \theta} \\ + \nu \left[\frac{\partial^2 u_\theta}{\partial R^2} + \frac{1}{R^2} \frac{\partial^2 u_\theta}{\partial \theta^2} + \frac{\partial^2 u_\theta}{\partial z^2} + \frac{1}{R} \frac{\partial u_\theta}{\partial R} + \frac{2}{R^2} \frac{\partial u_R}{\partial \theta} - \frac{u_\theta}{R^2} \right] + \frac{\partial \Phi}{\partial \theta} \end{aligned}$$

NOTE: There are several terms in the above expression that may seem ‘surprising’. The important thing to remember in writing down the equations in curvi-linear

coordinates is that operators can also act on unit-direction vectors. For example, the θ -component of $\nabla^2 \vec{u}$ is NOT $\nabla^2 u_\theta$. That is because the operator ∇^2 acts on $u_R \vec{e}_R + u_\theta \vec{e}_\theta + u_z \vec{e}_z$, and the directions of $\vec{e}_R$ and $\vec{e}_\theta$ depend on position! The same holds for the convective operator $(\vec{u} \cdot \nabla) \vec{u}$. The full expressions for both cylindrical and spherical coordinates are written out in Appendix D.

Setting all the terms containing $\partial/\partial\theta$ and/or u_z to zero, the Navier-Stokes equation simplifies considerably to

$$\rho \left[\frac{\partial u_\theta}{\partial t} + u_R \frac{\partial u_\theta}{\partial R} + \frac{u_R u_\theta}{R} \right] = \mu \left[\frac{\partial^2 u_\theta}{\partial R^2} + \frac{\partial^2 u_\theta}{\partial z^2} + \frac{1}{R} \frac{\partial u_\theta}{\partial R} - \frac{u_\theta}{R^2} \right]$$

where we have replaced the **kinetic viscosity**, ν , with $\mu = \nu\rho$.

Integrating over z and writing

$$\int_{-\infty}^{\infty} \rho dz = \Sigma$$

where Σ is the surface density, as well as neglecting variation of ν , u_R and u_θ with z (a reasonable approximation), the continuity and Navier-Stokes equation become

$$\begin{aligned} \frac{\partial \Sigma}{\partial t} + \frac{1}{R} \frac{\partial}{\partial R} (R \Sigma u_R) &= 0 \\ \Sigma \left(\frac{\partial u_\theta}{\partial t} + u_R \frac{\partial u_\theta}{\partial R} + \frac{u_R u_\theta}{R} \right) &= \mathcal{F}(\mu, R) \end{aligned}$$

where $\mathcal{F}(\mu, R)$ describes the various **viscous terms**.

Next we multiply the continuity equation by Ru_θ which we can then write as

$$\frac{\partial(\Sigma R u_\theta)}{\partial t} - \Sigma \frac{\partial(R u_\theta)}{\partial t} + \frac{\partial(\Sigma R u_R u_\theta)}{\partial R} - R \Sigma u_R \frac{\partial u_\theta}{\partial R} = 0$$

Adding this to R times the Navier-Stokes equation, and rearranging terms, yields

$$\frac{\partial(\Sigma R u_\theta)}{\partial t} + \frac{\partial(\Sigma R u_R u_\theta)}{\partial R} + \Sigma u_R u_\theta = \mathcal{G}(\mu, R)$$

where $\mathcal{G}(\mu, R) = R\mathcal{F}(\mu)$. Next we introduce the **angular frequency** $\Omega \equiv u_\theta/R$ which allows us to rewrite the above expression as

$$\boxed{\boxed{\frac{\partial(\Sigma R^2 \Omega)}{\partial t} + \frac{1}{R} \frac{\partial}{\partial R} (\Sigma R^3 \Omega u_R) = \mathcal{G}(\mu, R)}}$$

Note that $\Sigma R^2 \Omega = \Sigma R u_\theta$ is the *angular momentum per unit surface area*. Hence the above equation describes the evolution of angular momentum in the accretion disk. It is also clear, therefore, that $\mathcal{G}(\mu, R)$ must describe the **viscous torque** on the disk material, per unit surface area. To derive an expression for it, recall that

$$\mathcal{G}(\mu, R) = R \int dz \mu \left[\frac{\partial^2 u_\theta}{\partial R^2} + \frac{1}{R} \frac{\partial u_\theta}{\partial R} - \frac{u_\theta}{R^2} \right]$$

where we have ignored the $\partial^2 u_\theta / \partial z^2$ term which is assumed to be small. Using that $\mu = \nu \rho$ and that μ is independent of R and z (this is an assumption that underlies the Navier-Stokes equation from which we started) we have that

$$\mathcal{G}(\mu, R) = \nu R \Sigma \left[\frac{\partial^2 u_\theta}{\partial R^2} + \frac{1}{R} \frac{\partial u_\theta}{\partial R} - \frac{u_\theta}{R^2} \right]$$

Next we use that $u_\theta = \Omega R$ to write

$$\frac{\partial u_\theta}{\partial R} = \Omega + R \frac{d\Omega}{dR}$$

Substituting this in the above expression for $\mathcal{G}(\mu, R)$ yield

$$\mathcal{G}(\mu, R) = \nu \Sigma \left[R^2 \frac{d^2 \Omega}{dR^2} + 3R \frac{d\Omega}{dR} \right] = \frac{1}{R} \frac{\partial}{\partial R} \left(\nu \Sigma R^3 \frac{d\Omega}{dR} \right)$$

Substituting this expression for the viscous torque in the evolution equation for the angular momentum per unit surface density, we finally obtain the full set of equations that govern our thin accretion disk:

$$\begin{aligned} \frac{\partial}{\partial t} (\Sigma R^2 \Omega) + \frac{1}{R} \frac{\partial}{\partial R} (\Sigma R^3 \Omega u_R) &= \frac{1}{R} \frac{\partial}{\partial R} \left(\nu \Sigma R^3 \frac{d\Omega}{dR} \right) \\ \frac{\partial \Sigma}{\partial t} + \frac{1}{R} \frac{\partial}{\partial R} (R \Sigma u_R) &= 0 \\ \Omega &= \left(\frac{G M_\bullet}{R^3} \right)^{1/2} \end{aligned}$$

These three equations describe the dynamics of a thin, viscous accretion disk. The third equation indicates that we assume that the fluid is in Keplerian motion around the accreting object of mass $M_\bullet$. As discussed further below, this is a reasonable assumption as long as the accretion disk is thin.

Note that the non-zero u_R results in a **mass inflow rate**

$$\dot{M}(R) = -2\pi\Sigma R u_R$$

(a positive u_R reflects outwards motion).

Now let us consider a **steady accretion disk**. This implies that $\partial/\partial t = 0$ and that $\dot{M}(R) = \dot{M} \equiv \dot{M}_\bullet$ (the mass flux is constant throughout the disk, otherwise $\partial\Sigma/\partial t \neq 0$). In particular, the continuity equation implies that

$$R\Sigma u_R = C_1$$

Using the above expression for the mass inflow rate, we see that

$$C_1 = -\frac{\dot{M}_\bullet}{2\pi}$$

Similarly, for the Navier-Stokes equation, we have that

$$\Sigma R^3 \Omega u_R - \nu \Sigma R^3 \frac{d\Omega}{dR} = C_2$$

Using the boundary condition that at the radius of the accreting object, $R_\bullet$, the disk material must be dragged into rigid rotation (a no-slip boundary condition), which implies that $d\Omega/dR = 0$ at $R = R_\bullet$, we obtain that

$$C_2 = R_\bullet^2 \Omega_\bullet C_1 = -\frac{\dot{M}_\bullet}{2\pi} (G M_\bullet R_\bullet)^{1/2}$$

Substituting this in the above expression, and using that

$$\frac{d\Omega}{dR} = \frac{d}{dR} \left(\frac{G M_\bullet}{R^3} \right)^{1/2} = -\frac{3}{2} \frac{\Omega}{R}$$

we have that

$$\begin{aligned}\nu \Sigma &= -\frac{\dot{M}_\bullet}{2\pi} \left[R^2 \Omega + (G M_\bullet R_\bullet)^{1/2} \right] \left(R^3 \frac{d\Omega}{dR} \right)^{-1} \\ &= +\frac{\dot{M}_\bullet}{3\pi} \left[1 - \left(\frac{R_\bullet}{R} \right)^{1/2} \right]\end{aligned}$$

This shows that *the mass inflow rate and kinetic viscosity depend linearly on each other.*

The gravitational energy lost by the inspiraling material is converted into heat. This is done through **viscous dissipation**: viscosity robs the disk material of angular momentum which in turn causes it to spiral in.

We can work out the **rate of viscous dissipation** using

$$\mathcal{V} = \tau_{ij} \frac{\partial u_i}{\partial x_j}$$

where we have that the **deviatoric stress tensor** is

$$\tau_{ij} = \mu \left[\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} - \frac{2}{3} \delta_{ij} \frac{\partial u_k}{\partial x_k} \right]$$

(see Chapter 4). Note that the last term in the above expression vanishes because the fluid is incompressible, such that

$$\mathcal{V} = \mu \left[\left(\frac{\partial u_i}{\partial x_j} \right)^2 + \frac{\partial u_j}{\partial x_i} \frac{\partial u_i}{\partial x_j} \right]$$

(remember to apply the Einstein summation convention here!).

In our case, using that $\partial/\partial\theta = \partial/\partial z = 0$ and that $u_z = 0$, the only surviving terms are

$$\mathcal{V} = \mu \left[\left(\frac{\partial u_R}{\partial R} \right)^2 + \left(\frac{\partial u_\theta}{\partial R} \right)^2 + \frac{\partial u_R}{\partial R} \frac{\partial u_R}{\partial R} \right] = \mu \left[2 \left(\frac{\partial u_R}{\partial R} \right)^2 + \left(\frac{\partial u_\theta}{\partial R} \right)^2 \right]$$

If we make the reasonable assumption that $u_R \ll u_\theta$, we can ignore the first term, such that we finally obtain

$$\boxed{\mathcal{V} = \mu \left(\frac{\partial u_\theta}{\partial R} \right)^2 = \mu R^2 \left(\frac{d\Omega}{dR} \right)^2}$$

which expresses the viscous dissipation *per unit volume*. Note that there is no viscous dissipation if $d\Omega/dR = 0$, i.e., in the case of solid body rotation. This makes sense since in that case there is no shear in the disk.

As before, we now proceed by integrating over the z -direction, to obtain

$$\frac{dE}{dt} = \int \mu R^2 \left(\frac{d\Omega}{dR} \right)^2 dz = \nu \Sigma R^2 \left(\frac{d\Omega}{dR} \right)^2$$

Using our expression for $\nu\Sigma$ derived above, we can rewrite this as

$$\frac{dE}{dt} = \frac{\dot{M}_\bullet}{3\pi} R^2 \left[1 - \left(\frac{R_\bullet}{R} \right)^{1/2} \right] \left(\frac{d\Omega}{dR} \right)^2$$

Using once more that $d\Omega/dR = -(3/2)\Omega/R$, and integrating over the entire disk yields the accretion luminosity of a thin accretion disk:

$$\boxed{L_{\text{acc}} \equiv 2\pi \int_{R_\bullet}^{\infty} \frac{dE}{dt} R dR = \frac{G M_\bullet \dot{M}_\bullet}{2 R_\bullet}}$$

To put this in perspective, realize that the gravitation energy of mass m at radius $R_\bullet$ is $G M_\bullet m / R_\bullet$. Thus, L_{acc} is exactly half of the gravitational energy lost due to the inflow. This obviously begs the question where the other half went...The answer is simple; it is stored in kinetic energy at the ‘boundary’ radius $R_\bullet$ of the accreting flow.

We end our discussion on accretion disks with a few words of caution. First of all, our entire derivation is only valid for a thin accretion disk. In a thin disk, the

pressure in the disk must be small (otherwise it would puff up). This means that the $\partial P/\partial R$ term in the R -component of the Navier-Stokes equation is small compared to $\partial\Phi/\partial R = GM/R^2$. This in turn implies that the gas will indeed be moving on Keplerian orbits, as we have assumed. If the accretion disk is thick, the situation is much more complicated, something that will not be covered in this course.

Finally, let us consider the time scale for accretion. As we have seen above, the energy loss rate per unit surface area is

$$\nu \Sigma R^2 \left(\frac{d\Omega}{dR} \right)^2 = \frac{9}{4} \nu \frac{G M_{\bullet}}{R^3}$$

We can compare this with the gravitational potential energy of disk material per unit surface area, which is

$$E = \frac{G M_{\bullet} \Sigma}{R}$$

This yields an accretion time scale

$$t_{\text{acc}} \equiv \frac{E}{dE/dt} = \frac{4}{9} \frac{R^2}{\nu} \sim \frac{R^2}{\nu}$$

To estimate this time-scale, we first estimate the molecular viscosity. Recall that $\nu \propto \lambda_{\text{mfp}} v$ with v a typical velocity of the fluid particles. In virtually all cases encountered in astrophysics, we have that the size of the accretion disk, R , is many, many orders of magnitude larger than λ_{mfp} . As a consequence, the corresponding t_{acc} easily exceeds the Hubble time!

The conclusion is that **molecular viscosity** is way too small to result in any significant accretion in objects of astrophysical size. Hence, other source of viscosity are required, which is a topic of ongoing discussion in the literature. Probably the most promising candidates are turbulence (in different forms), and the magneto-rotational instability (MRI). Given the uncertainties involved, it is common practice to simply write

$$\nu = \alpha \frac{P}{\rho} \frac{1}{R} \left(\frac{d\Omega}{dR} \right)^{-1}$$

where α is a ‘free parameter’. A thin accretion disk modelled this way is often called an **alpha-accretion disk**. If you wonder what the origin is of the above expression;



Figure 8: *Image of the central region of NGC 4261 taken with the Hubble Space Telescope. It reveals a $\sim 100\text{pc}$ scale disk of dust and gas, which happens to be perpendicular to a radio jet that emerges from this galaxy. This is an alledged ‘accretion disk’ supplying fuel to the central black hole in this galaxy.*

it simply comes from assuming that the only non-vanishing off-diagonal term of the stress tensor is taken to be αP (where P is the value along the diagonal of the stress tensor).

CHAPTER 10

Turbulence

Non-linearity: The Navier-Stokes equation is non-linear. This non-linearity arises from the convective (material) derivative term

$$\vec{u} \cdot \nabla \vec{u} = \frac{1}{2} \nabla u^2 - \vec{u} \times \vec{w}$$

which describes the "inertial acceleration" and is ultimately responsible for the origin of the **chaotic character** of many flows and of **turbulence**. Because of this non-linearity, we cannot say whether a solution to the Navier-Stokes equation with nice and smooth initial conditions will remain nice and smooth for all time (at least not in 3D).

Laminar flow: occurs when a fluid flows in parallel layers, without lateral mixing (no cross currents perpendicular to the direction of flow). It is characterized by high momentum diffusion and low momentum convection.

Turbulent flow: is characterized by chaotic and stochastic property changes. This includes low momentum diffusion, high momentum convection, and rapid variation of pressure and velocity in space and time.

The Reynold's number: In order to gauge the importance of viscosity for a fluid, it is useful to compare the ratio of the inertial acceleration ($\vec{u} \cdot \nabla \vec{u}$) to the viscous acceleration ($\nu [\nabla^2 \vec{u} + \frac{1}{3} \nabla(\nabla \cdot \vec{u})]$). This ratio is called the Reynold's number, $\mathcal{R}$, and can be expressed in terms of the typical velocity scale $U \sim |\vec{u}|$ and length scale $L \sim 1/\nabla$ of the flow, as

$$\mathcal{R} = \left| \frac{\vec{u} \cdot \nabla \vec{u}}{\nu [\nabla^2 \vec{u} + \frac{1}{3} \nabla(\nabla \cdot \vec{u})]} \right| \sim \frac{U^2/L}{\nu U/L^2} = \frac{U L}{\nu}$$

If $\mathcal{R} \gg 1$ then viscosity can be ignored (and one can use the Euler equations to describe the flow). However, if $\mathcal{R} \ll 1$ then viscosity is important.

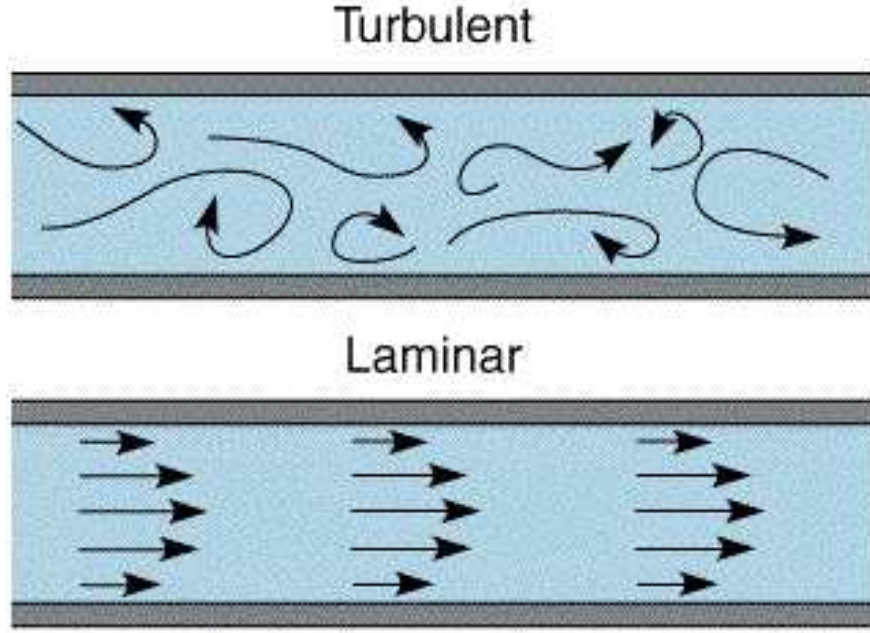


Figure 9: *Illustration of laminar vs. turbulent flow.*

Similarity: Flows with the same Reynold's number are similar. This is evident from rewriting the Navier-Stokes equation in terms of the following dimensionless variables

$$\tilde{u} = \frac{\vec{u}}{U} \quad \tilde{x} = \frac{\vec{x}}{L} \quad \tilde{t} = t \frac{U}{L} \quad \tilde{p} = \frac{P}{\rho U^2} \quad \tilde{\Phi} = \frac{\Phi}{U^2} \quad \tilde{\nabla} = L \nabla$$

This yields (after multiplying the Navier-Stokes equation with L/U^2):

$$\boxed{\boxed{\frac{\partial \tilde{u}}{\partial \tilde{t}} + \tilde{u} \cdot \tilde{\nabla} \tilde{u} + \tilde{\nabla} \tilde{p} + \tilde{\nabla} \tilde{\Phi} = \frac{1}{\mathcal{R}} \left[\tilde{\nabla}^2 \tilde{u} + \frac{1}{3} \tilde{\nabla} (\tilde{\nabla} \cdot \tilde{u}) \right]}}$$

which shows that the form of the solution depends only on $\mathcal{R}$. This principle is extremely powerful as it allows one to making scale models (i.e., when developing airplanes, cars etc). NOTE: the above equation is only correct for an incompressible fluid, i.e., a fluid that obeys $\nabla \rho = 0$. If this is not the case the term $\tilde{P}(\nabla \rho / \rho)$ needs to be added at the rhs of the equation, braking its scale-free nature.

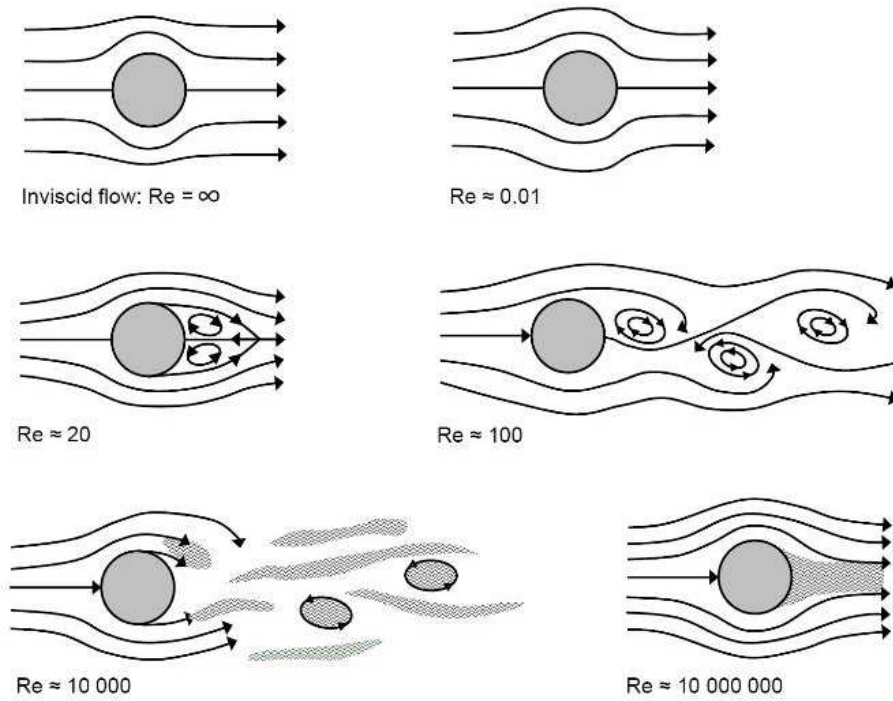


Figure 10: *Illustration of flows at different Reynolds number.*

As a specific example, consider fluid flow past a cylinder of diameter L :

- $\mathcal{R} \ll 1$: "creeping flow". In this regime the flow is viscously dominated and (nearly) symmetric upstream and downstream. The inertial acceleration ($\vec{u} \cdot \nabla \vec{u}$) can be neglected, and the flow is (nearly) time-reversible.
- $\mathcal{R} \sim 1$: Slight asymmetry develops
- $10 \leq \mathcal{R} \leq 41$: Separation occurs, resulting in two counter-rotating vortices in the wake of the cylinder. The flow is still steady and laminar, though.
- $41 \leq \mathcal{R} \leq 10^3$: "von Kármán vortex street"; **unsteady laminar flow** with counter-rotating vortices shed periodically from the cylinder (see Fig. 11). Even at this stage the flow is still 'predictable'.
- $\mathcal{R} > 10^3$: vortices are unstable, resulting in a **turbulent wake** behind the cylinder that is 'unpredictable'.

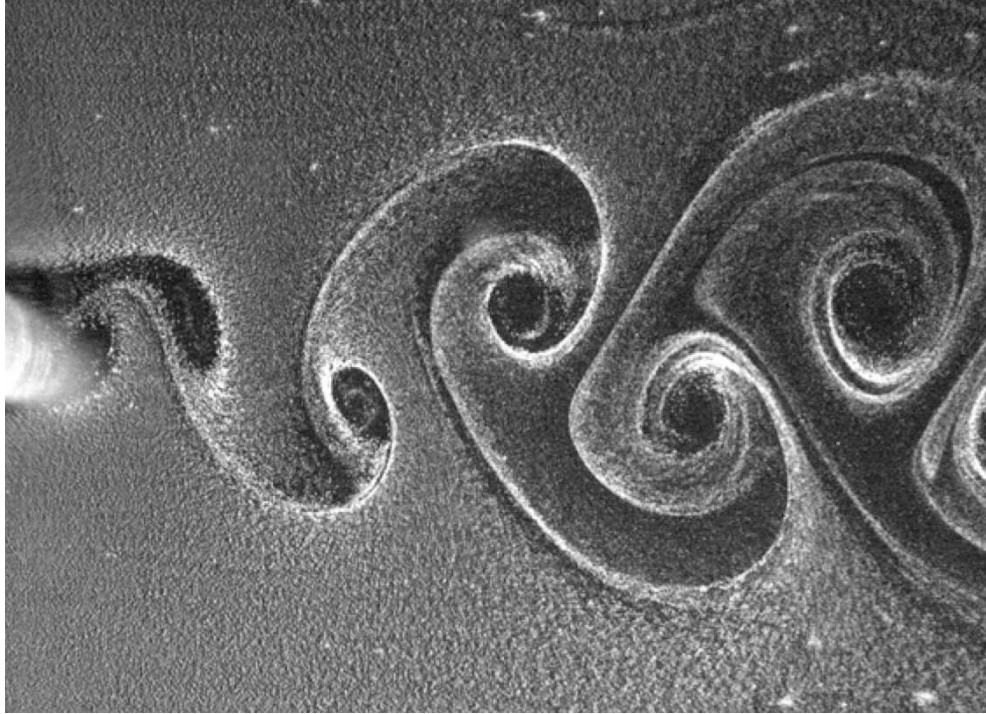


Figure 11: *The image shows the von Kármán Vortex street behind a 6.35 mm diameter circular cylinder in water at Reynolds number of 168. The visualization was done using hydrogen bubble technique. Credit: Sanjay Kumar & George Laughlin, Department of Engineering, The University of Texas at Brownsville*

The following movie shows a $\mathcal{R} = 250$ flow past a cylinder. Initially one can witness separation, and the creation of two counter-rotating vortices, which then suddenly become ‘unstable’, resulting in the von Kármán vortex street:

<http://www.youtube.com/watch?v=IDeGDFZSYo8>

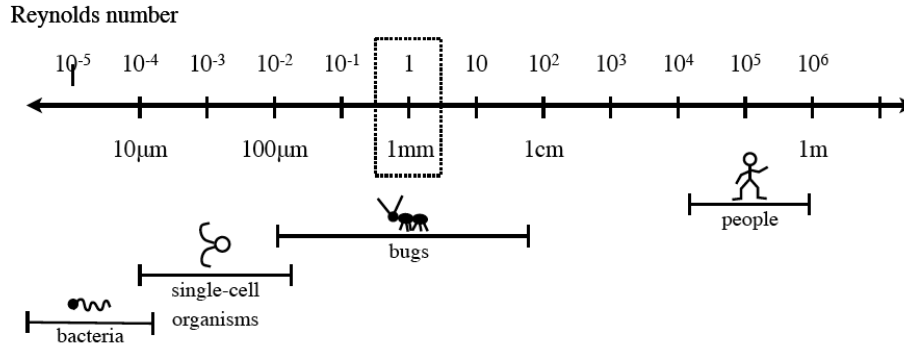


Figure 12: *Typical Reynolds numbers for various biological organisms. Reynolds numbers are estimated using the length scales indicated, the rule-of-thumb in the text, and material properties of water.*

Locomotion at Low-Reynolds number: Low Reynolds number corresponds to high kinetic viscosity *for a given U and L* . In this regime of ‘creeping flow’ the flow past an object is (nearly) time-reversible. Imagine trying to move (swim) in a highly viscous fluid (take honey as an example). If you try to do so by executing time-symmetric movements, you will not move. Instead, you need to think of a symmetry-breaking solution. Nature has found many solutions for this problem. If we make the simplifying “rule-of-thumb” assumption that an animal of size L meters moves roughly at a speed of $U = L$ meters per second (yes, this is very, very rough, but an ant *does* move close to 1 mm/s, and a human at roughly 1 m/s), then we have that $\mathcal{R} = UL/\nu \simeq L^2/\nu$. Hence, with respect to a fixed substance (say water, for which $\nu \sim 10^{-2}\text{cm}^2/\text{s}$), smaller organisms move at lower Reynolds number (effectively in a fluid of higher viscosity). Scaling down from a human to bacteria and single-cell organisms, the motion of the latter in water has $\mathcal{R} \sim 10^{-5} - 10^{-2}$ (see Fig. 12). Understanding the locomotion of these organisms is a fascinating sub-branch of bio-physics.

Boundary Layers: Even when $\mathcal{R} \gg 1$, viscosity always remains important in thin boundary layers adjacent to any solid surface. This boundary layer must exist in order to satisfy the **no-slip boundary condition**. If the Reynolds number exceeds a critical value, the boundary layer becomes turbulent. Turbulent layers and their associated turbulent wakes exert a much bigger drag on moving bodies than their laminar counterparts.

Momentum Diffusion & Reynolds stress: This gives rise to an interesting phenomenon. Consider flow through a pipe. If you increase the viscosity (i.e., decrease $\mathcal{R}$), then it requires a larger force to achieve a certain flow rate (think of how much harder it is to push honey through a pipe compared to water). However, this trend is not monotonic. For sufficiently low viscosity (large $\mathcal{R}$), one finds that the trend reverses, and that it becomes harder again to push the fluid through the pipe. This is a consequence of turbulence, which causes **momentum diffusion** within the flow, which acts very much like viscosity. However, this momentum diffusion is not due to the **viscous stress tensor**, τ_{ij} , but rather to the **Reynolds stress tensor** R_{ij} . To understand the ‘origin’ of the Reynolds stress tensor, consider the following:

For a turbulent flow, $\vec{u}(t)$, it is advantageous to decompose each component of $\vec{u}$ into a ‘mean’ component, $\bar{u}_i$, and a ‘fluctuating’ component, u'_i , according to

$$u_i = \bar{u}_i + u'_i$$

This is known as the **Reynolds decomposition**. The ‘mean’ component can be a time-average, a spatial average, or an ensemble average, depending on the detailed characteristics of the flow. Note that this is reminiscent of how we decomposed the microscopic velocities of the fluid particles in a ‘mean’ velocity (describing the fluid elements) and a ‘random, microscopic’ velocity ($\vec{v} = \vec{u} + \vec{w}$).

Substituting this into the Navier-Stokes equation, and taking the average of that, we obtain

$$\frac{\partial \bar{u}_i}{\partial t} + \bar{u}_j \frac{\partial \bar{u}_i}{\partial x_j} = \frac{1}{\rho} \frac{\partial}{\partial x_j} [\bar{\sigma}_{ij} - \rho \overline{u'_i u'_j}]$$

where, for simplicity, we have ignored gravity (the $\nabla\Phi$ -term). This equation looks identical to the Navier-Stokes equation (in absence of gravity), except for the $-\rho \overline{u'_i u'_j}$ term, which is what we call the Reynolds stress tensor:

$$R_{ij} = -\rho \overline{u'_i u'_j}$$

Note that $\overline{u'_i u'_j}$ means the same averaging (time, space or ensemble) as above, but now for the product of u'_i and u'_j . Note that $\overline{u'_i} = 0$, by construction. However, the expectation value for the product of u'_i and u'_j is generally not. As is evident from the equation, the **Reynolds stresses** (which reflect momentum diffusion due to turbulence) act in exactly the same way as the **viscous stresses**. However, they are only present when the flow is turbulent.

Note also that the Reynolds stress tensor is related to the **two-point correlation tensor**

$$\xi_{ij}(\vec{r}) \equiv \overline{u'_i(\vec{x}, t) u'_j(\vec{x} + \vec{r}, t)}$$

in the sense that $R_{ij} = \xi_{ij}(0)$. At large separations, $\vec{r}$, the fluctuating velocities will be uncorrelated so that $\lim_{r \rightarrow \infty} \xi_{ij} = 0$. But on smaller scales the fluctuating velocities will be correlated, and there will be a ‘characteristic’ scale associated with these correlations, called the **correlation length**.

Turbulence: Turbulence is still considered as one of the last ”unsolved problems of classical physics” [Richard Feynman]. What we technically mean by this is that we do not yet know how to calculate $\xi_{ij}(\vec{r})$ (and higher order correlation functions, like the three-point, four-point, etc) in a particular situation from a fundamental theory. Salmon (1998) nicely sums up the challenge of defining **turbulence**:

Every aspect of turbulence is controversial. Even the definition of fluid turbulence is a subject of disagreement. However, nearly everyone would agree with some elements of the following description:

- Turbulence requires the presence of vorticity; irrotational flow is smooth and steady to the extent that the boundary conditions permit.
- Turbulent flow has a complex structure, involving a broad range of space and time scales.
- Turbulent flow fields exhibit a high degree of apparent randomness and disorder. However, close inspection often reveals the presence of embedded coherent flow structures

- Turbulent flows have a high rate of viscous energy dissipation.
- Advected tracers are rapidly mixed by turbulent flows.

However, one further property of turbulence seems to be more fundamental than all of these because it largely explains why turbulence demands a statistical treatment...turbulence is chaotic.

The following is a brief, qualitative description of turbulence:

Turbulence kicks in at sufficiently high Reynolds number (typically $\mathcal{R} > 10^3 - 10^4$). Turbulent flow is characterized by irregular and seemingly random motion. Large vortices (called **eddies**) are created. These contain a large amount of kinetic energy. Due to **vortex stretching** these eddies are stretched thin until they ‘break up’ in smaller eddies. This results in a **cascade** in which the turbulent energy is transported from large scales to small scales. This cascade is largely **inviscid**, conserving the total turbulent energy. However, once the length scale of the eddies becomes comparable to the mean free path of the particles, the energy is dissipated; the kinetic energy associated with the eddies is transformed into internal energy. The scale at which this happens is called the **Kolmogorov length scale**. The length scales between the scale of turbulence ‘injection’ and the Kolomogorov length scale at which it is dissipated is called the **inertial range**. Over this inertial range turbulence is believed/observed to be scale invariant. The ratio between the injection scale, L , and the dissipation scale, l , is proportional to the **Reynolds number** according to $L/l \propto \mathcal{R}^{3/4}$. Hence, two turbulent flows that look similar on large scales (comparable L), will dissipate their energies on different scales, l , if their Reynolds numbers are different.

Molecular clouds: an example of turbulence in astrophysics are molecular clouds. These are gas clouds of masses $10^5 - 10^6 M_\odot$, densities $n_H \sim 100 - 500 \text{ cm}^{-3}$, and temperatures $T \sim 10\text{K}$. They consist mainly of molecular hydrogen and are the main sites of **star formation**. Observations show that their velocity linewidths are $\sim 6 - 10 \text{ km/s}$, which is much higher than their sound speed ($c_s \sim 0.2 \text{ km/s}$). Hence, they are supported against (gravitational) collapse by **supersonic turbulence**. On small scales, however, the turbulent motions compress the gas to high enough densities that stars can form. A numerical simulation of a molecular cloud with supersonic turbulence is available here:

<http://www.youtube.com/watch?v=3z9ZKAkbMhY>

CHAPTER 11

Sound Waves

If a (compressible) fluid in equilibrium is perturbed, and the perturbation is sufficiently small, the perturbation will propagate through the fluid as a **sound wave** (aka acoustic wave), which is a mechanical, longitudinal wave (i.e., a displacement in the same direction as that of propagation).

If the perturbation is small, we may assume that the velocity gradients are so small that viscous effects are negligible (i.e., we can set $\nu = 0$). In addition, we assume that the time scale for conductive heat transport is large, so that energy exchange due to conduction can also safely be ignored. In the absence of these dissipative processes, the wave-induced changes in gas properties are **adiabatic**.

Before proceeding, let us examine the **Reynold's number** of a (propagating) sound wave. Using that $\mathcal{R} = U L / \nu$, and setting $U = c_s$ (the typical velocity involved is the sound speed, to be defined below), $L = \lambda$ (the characteristic scale of the flow is the wavelength of the acoustic wave), we have that $\mathcal{R} = \lambda c_s / \nu$. Using the expressions for the viscosity $\mu = \nu \rho$ from the constitutive relations in Chapter 5, we see that $\nu \propto \lambda_{\text{mfp}} c_s$. Hence, we have that

$$\mathcal{R} \equiv \frac{U L}{\nu} \propto \frac{\lambda}{\lambda_{\text{mfp}}}$$

Thus, as long as the wave-length of the acoustic wave is much larger than the mean-free path of the fluid particles, we have that the Reynolds number is large, and thus that viscosity and conduction can be ignored.

Let $(\rho_0, P_0, \vec{u}_0)$ be a **uniform, equilibrium solution** of the Euler fluid equations (i.e., ignore viscosity). Also, in what follows we will ignore gravity (i.e., $\nabla \Phi = 0$).

Uniformity implies that $\nabla \rho_0 = \nabla P_0 = \nabla \vec{u}_0 = 0$. In addition, since the only allowed motion is uniform motion of the entire system, we can always use a Galilean coordinate transformation so that $\vec{u}_0 = 0$, which is what we adopt in what follows.

Substitution into the continuity and momentum equations, one obtains that $\partial\rho_0/\partial t = \partial\vec{u}_0/\partial t = 0$, indicative of an **equilibrium** solution as claimed.

Perturbation Analysis: Consider a small perturbation away from the above equilibrium solution:

$$\begin{aligned}\rho_0 &\rightarrow \rho_0 + \rho_1 \\ P_0 &\rightarrow P_0 + P_1 \\ \vec{u}_0 &\rightarrow \vec{u}_0 + \vec{u}_1 = \vec{u}_1\end{aligned}$$

where $|\rho_1/\rho_0| \ll 1$, $|P_1/P_0| \ll 1$ and $\vec{u}_1$ is small (compared to the sound speed, to be derived below).

Substitution in the **continuity** and **momentum** equations yields

$$\begin{aligned}\frac{\partial(\rho_0 + \rho_1)}{\partial t} + \nabla(\rho_0 + \rho_1)\vec{u}_1 &= 0 \\ \frac{\partial\vec{u}_1}{\partial t} + \vec{u}_1 \cdot \nabla\vec{u}_1 &= -\frac{\nabla(P_0 + P_1)}{(\rho_0 + \rho_1)}\end{aligned}$$

which, using that $\nabla\rho_0 = \nabla P_0 = \nabla\vec{u}_0 = 0$ reduces to

$$\begin{aligned}\frac{\partial\rho_1}{\partial t} + \rho_0\nabla\vec{u}_1 + \nabla(\rho_1\vec{u}_1) &= 0 \\ \frac{\partial\vec{u}_1}{\partial t} + \frac{\rho_1}{\rho_0}\frac{\partial\vec{u}_1}{\partial t} + \vec{u}_1 \cdot \nabla\vec{u}_1 + \frac{\rho_1}{\rho_0}\vec{u}_1 \cdot \nabla\vec{u}_1 &= -\frac{\nabla P_1}{\rho_0}\end{aligned}$$

The latter follows from first multiplying the momentum equations with $(\rho_0 + \rho_1)/\rho_0$. Note that we don't need to consider the **energy equation**; this is because (i) we have assumed that conduction is negligible, and (ii) the disturbance is adiabatic (meaning $dQ = 0$, and there is thus no heating or cooling).

Next we **linearize** these equations, which means we use that the perturbed values are all small such that terms that contain products of two or more of these quantities are always negligible compared to those that contain only one such quantity. Hence, the above equations reduce to

$$\begin{aligned}\frac{\partial\rho_1}{\partial t} + \rho_0\nabla\vec{u}_1 &= 0 \\ \frac{\partial\vec{u}_1}{\partial t} + \frac{\nabla P_1}{\rho_0} &= 0\end{aligned}$$

These equations describe the evolution of perturbations in an **inviscid** and **uniform** fluid. As always, these equations need an additional equation for closure. As mentioned above, we don't need the **energy equation**: instead, we can use that the flow is adiabatic, which implies that $P \propto \rho^\gamma$.

Using Taylor series expansion, we then have that

$$P(\rho_0 + \rho_1) = P(\rho_0) + \left(\frac{\partial P}{\partial \rho} \right)_0 \rho_1 + \mathcal{O}(\rho_1^2)$$

where we have used $(\partial P / \partial \rho)_0$ as shorthand for the partial derivative of $P(\rho)$ at $\rho = \rho_0$. And since the flow is isentropic, we have that the partial derivative is for constant entropy. Using that $P(\rho_0) = P_0$ and $P(\rho_0 + \rho_1) = P_0 + P_1$, we find that, when linearized,

$$P_1 = \left(\frac{\partial P}{\partial \rho} \right)_0 \rho_1$$

Note that $P_1 \neq P(\rho_1)$; rather P_1 is the perturbation in pressure associated with the perturbation ρ_1 in the density.

Substitution in the fluid equations of our perturbed quantities yields

$$\begin{aligned} \frac{\partial \rho_1}{\partial t} + \rho_0 \nabla \cdot \vec{u}_1 &= 0 \\ \frac{\partial \vec{u}_1}{\partial t} + \left(\frac{\partial P}{\partial \rho} \right)_0 \frac{\nabla \rho_1}{\rho_0} &= 0 \end{aligned}$$

Taking the partial time derivative of the above **continuity** equation, and using that $\partial \rho_0 / \partial t = 0$, gives

$$\frac{\partial^2 \rho_1}{\partial t^2} + \rho_0 \nabla \cdot \frac{\partial \vec{u}_1}{\partial t} = 0$$

Substituting the above **momentum equation**, and realizing that $(\partial P / \partial \rho)_0$ is a constant, then yields

$$\boxed{\boxed{\frac{\partial^2 \rho_1}{\partial t^2} - \left(\frac{\partial P}{\partial \rho} \right)_0 \nabla^2 \rho_1 = 0}}$$

which we recognize as a **wave equation**, whose solution is a **plane wave**:

$$\rho_1 \propto e^{i(\vec{k} \cdot \vec{x} - \omega t)}$$

with $\vec{k}$ the **wavevector**, $k = |\vec{k}| = 2\pi/\lambda$ the **wavenumber**, λ the **wavelength**, $\omega = 2\pi\nu$ the **angular frequency**, and ν the **frequency**.

To gain some insight, consider the 1D case: $\rho_1 \propto e^{i(kx-\omega t)} \propto e^{ik(x-v_p t)}$, where we have defined the **phase velocity** $v_p \equiv \omega/k$. This is the velocity with which the wave pattern propagates through space. For our perturbation of a compressible fluid, this phase velocity is called the **sound speed**, c_s . Substituting the solution $\rho_1 \propto e^{i(kx-\omega t)}$ into the wave equation, we see that

$$c_s = \frac{\omega}{k} = \sqrt{\left(\frac{\partial P}{\partial \rho}\right)_s}$$

where we have made it explicit that the flow is assumed to be isentropic. Note that the partial derivative is for the unperturbed medium. This sound speed is sometimes called the **adiabatic speed of sound**, to emphasize that it relies on the assumption of an adiabatic perturbation. If the fluid is an ideal gas, then

$$c_s = \sqrt{\gamma \frac{k_B T}{\mu m_p}}$$

which shows that the adiabatic sound speed of an ideal fluid increases with temperature.

We can repeat the above derivation by relaxing the assumption of isentropic flow, and assuming instead that (more generally) the flow is **polytropic**. In that case, $P \propto \rho^\Gamma$, with Γ the **polytropic index** (Note: a polytropic EoS is an example of a **barotropic** EoS). The only thing that changes is that now the sound speed becomes

$$c_s = \sqrt{\frac{\partial P}{\partial \rho}} = \sqrt{\Gamma \frac{P}{\rho}}$$

which shows that the sound speed is larger for a stiffer EoS (i.e., a larger value of Γ).

Note also that, for our barotropic fluid, the sound speed is independent of ω . This implies that all waves move equally fast; the shape of a wave packet is preserved

as it moves. We say that an ideal (inviscid) fluid with a barotropic EoS is a **non-dispersive medium**.

To gain further insight, let us look once more at the (1D) solution for our perturbation:

$$\rho_1 \propto e^{i(kx - \omega t)} \propto e^{ikx} e^{-i\omega t}$$

Recalling **Euler’s formula** ($e^{i\theta} = \cos \theta + i \sin \theta$), we see that:

- The e^{ikx} part describes a periodic, spatial oscillation with wavelength $\lambda = 2\pi/k$.
- The $e^{-i\omega t}$ part describes the time evolution:
 - If ω is **real**, then the solution describes a **sound wave** which propagates through space with a sound speed c_s .
 - If ω is **imaginary** then the perturbation is either exponentially growing (‘unstable’) or decaying (‘damped’) with time.

We will return to this in Chapter 15, when we discuss the **Jeans stability criterion**.

As discussed above, acoustic waves result from disturbances in a compressible fluid. These disturbances may arise from objects being moved through the fluid. However, sound waves can also be sourced by fluid motions themselves. A familiar example is the noise from jet-engines; the noise emanates from the turbulent wake created by engines. In astrophysics, turbulence will also typically create sound waves. In general these sound waves will not have an important impact on the physics. A potential exception is the heating of the ICM by sound waves created by turbulent wakes created by AGN feedback.

CHAPTER 12

Shocks

When discussing sound waves in the previous chapter, we considered small (linear) perturbations. In this Chapter we consider the case in which the perturbations are large (non-linear). Typically, a large disturbance results in an abrupt **discontinuity** in the fluid, called a **shock**. Note: not all discontinuities are shocks, but all shocks are discontinuities.

Consider a **polytropic** EoS:

$$P = P_0 \left(\frac{\rho}{\rho_0} \right)^\Gamma$$

The sound speed is given by

$$c_s = \left(\frac{\partial P}{\partial \rho} \right)^{1/2} = \sqrt{\Gamma \frac{P}{\rho}} = c_{s,0} \left(\frac{\rho}{\rho_0} \right)^{(\Gamma-1)/2}$$

If $\Gamma = 1$, i.e., the EoS is **isothermal**, then the sound speed is a constant, independent of density or pressure. However, if $\Gamma \neq 1$, then the sound speed varies with the local density. An important example, often encountered in (astro)physics is the **adiabatic** EoS, for which $\Gamma = \gamma$ ($\gamma = 5/3$ for a mono-atomic gas). In that case we have that c_s increases with density (and pressure, and temperature).

In our discussion of sound waves (Chapter 13), we used perturbation theory, in which we neglected the $\vec{u}_1 \cdot \nabla \vec{u}_1$ term. However, when the perturbations are not small, this term is no longer negligible, and causes non-linearities to develop. The most important of those, is the fact that the sound speed itself varies with density (as we have seen above). This implies that the wave-form of the acoustic wave changes with time; the wave-crest is moving faster than the wave-trough, causing an overall steepening of the wave-form. This steepening continues until the wave-crest tries to overtake the wave-trough, which is not allowed, giving rise to a shock front (see Fig. 13).

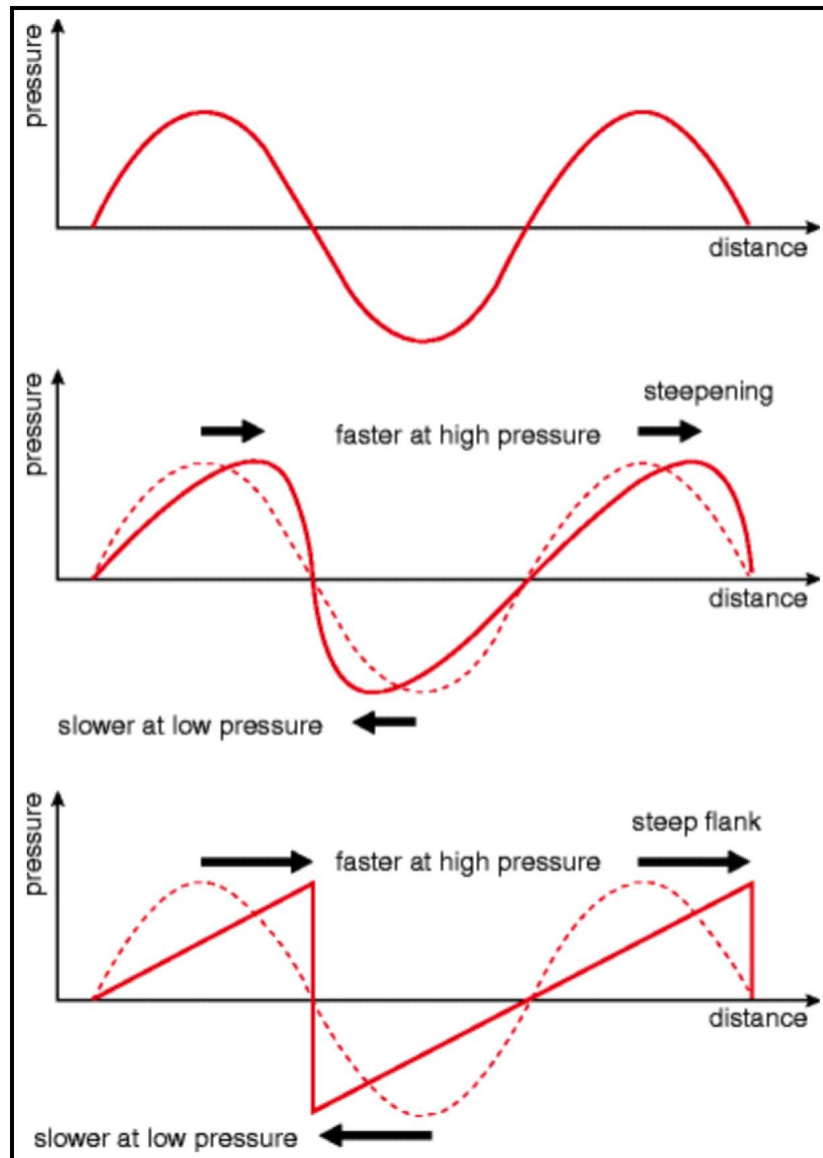


Figure 13: *The steepening of a sound wave into a shock due to non-linearity (i.e., sound speed depends on density).*

Mach Number: if v is the flow speed of the fluid, and c_s is the sound speed, then the Mach number of the flow is defined as

$$\mathcal{M} = \frac{v}{c_s}$$

Note: *simply accelerating a flow to supersonic speeds does **not** necessarily generate a shock.* Shocks only arise when an obstruction in the flow causes a deceleration of fluid moving at supersonic speeds. The reason is that disturbances cannot propagate upstream, so that the flow cannot ‘adjust itself’ to the obstacle because there is no way of propagating a signal (which always goes at the sound speed) in the upstream direction. Consequently, the flow remains undisturbed until it hits the obstacle, resulting in a discontinuous change in flow properties; a shock.

Structure of a Shock: Fig. 14 shows the structure of a planar shock. The shock has a finite, non-zero width (typically a few mean-free paths of the fluid particles), and separates the ‘up-stream’, pre-shocked gas, from the ‘down-stream’, shocked gas.

For reasons that will become clear in what follows, it is useful to split the downstream region in two sub-regions; one in which the fluid is out of thermal equilibrium, with net cooling $\mathcal{L} > 0$, and, further away from the shock, a region where the downstream gas is (once again) in thermal equilibrium (i.e., $\mathcal{L} = 0$). If the transition between these two sub-regions falls well outside the shock (i.e., if $x_3 \gg x_2$) the shock is said to be **adiabatic**. In that case, we can derive a relation between the upstream (pre-shocked) properties (ρ_1, P_1, T_1, u_1) and the downstream (post-shocked) properties (ρ_2, P_2, T_2, u_2) ; these relations are called the **Rankine-Hugoniot jump conditions**. Linking the properties in region three (ρ_3, P_3, T_3, u_3) to those in the pre-shocked gas is in general not possible, except in the case where $T_3 = T_1$. In this case one may consider the shock to be **isothermal**.

Rankine-Hugoniot jump conditions: We now derive the relations between the up- and down-stream quantities, under the assumption that the shock is adiabatic. Consider a rectangular volume V that encloses part of the shock; it has a thickness $dx > (x_2 - x_1)$ and is centered in the x -direction on the middle of shock. At fixed x the volume is bounded by an area A . If we ignore variations in ρ and $\vec{u}$ in the y -

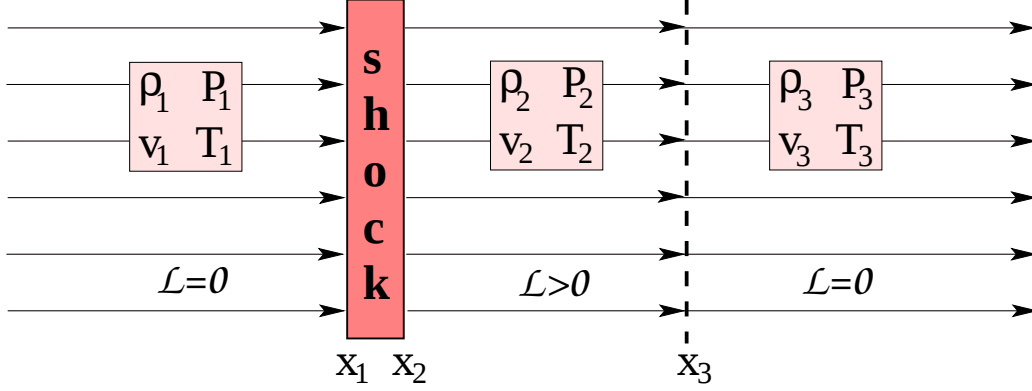


Figure 14: *Structure of a planar shock.*

and z -directions, the **continuity equation** becomes

$$\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x}(\rho u_x) = 0$$

If we integrate this equation over our volume V we obtain

$$\begin{aligned} & \int \int \int \frac{\partial \rho}{\partial t} dx dy dz + \int \int \int \frac{\partial}{\partial x}(\rho u_x) dx dy dz = 0 \\ \Leftrightarrow & \frac{\partial}{\partial t} \int \rho dx dy dz + A \int \frac{\partial}{\partial x}(\rho u_x) dx = 0 \\ \Leftrightarrow & \frac{\partial M}{\partial t} + \int d(\rho u_x) = 0 \end{aligned}$$

Since there is no mass accumulation in the shock, and mass does not disappear in the shock, we have that

$$\rho u_x|_{+dx/2} = \rho u_x|_{-dx/2}$$

In terms of the upstream (index 1) and downstream (index 2) quantities:

$$\boxed{\rho_1 u_1 = \rho_2 u_2}$$

This equation describes **mass conservation** across a shock.

The momentum equation in the x -direction, ignoring viscosity, is given by

$$\frac{\partial}{\partial t}(\rho u_x) = -\frac{\partial}{\partial x}(\rho u_x u_x + P) - \rho \frac{\partial \Phi}{\partial x}$$

Integrating this equation over V and ignoring any gradient in Φ across the shock, we obtain

$$\boxed{\boxed{\rho_1 u_1^2 + P_1 = \rho_2 u_2^2 + P_2}}$$

This equation describes how the shock **converts ram pressure into thermal pressure**.

Finally, applying the same to the **energy equation** under the assumption that the shock is adiabatic (i.e., $dQ/dt = 0$), one finds that $(E + P)u$ has to be the same on both sides of the shock, i.e.,

$$\left[\frac{1}{2}u^2 + \Phi + \varepsilon + \frac{P}{\rho} \right] \rho u = \text{constant}$$

We have already seen that ρu is constant. Hence, if we once more ignore gradients in Φ across the shock, we obtain that

$$\boxed{\boxed{\frac{1}{2}u_1^2 + \varepsilon_1 + P_1/\rho_1 = \frac{1}{2}u_2^2 + \varepsilon_2 + P_2/\rho_2}}$$

This equation describes how the shock **converts kinetic energy into enthalpy**. Qualitatively, a shock converts an ordered flow upstream into a disordered (hot) flow downstream.

The three equations in the rectangular boxes are known as the **Rankine-Hugoniot (RH) jump conditions for an adiabatic shock**. Using straightforward but tedious algebra, these RH jump conditions can be written in a more useful form using the **Mach number** $\mathcal{M}_1$ of the upstream gas:

$$\begin{aligned} \frac{\rho_2}{\rho_1} &= \frac{u_1}{u_2} = \left[\frac{1}{\mathcal{M}_1^2} + \frac{\gamma-1}{\gamma+1} \left(1 - \frac{1}{\mathcal{M}_1^2} \right) \right]^{-1} \\ \frac{P_2}{P_1} &= \frac{2\gamma}{\gamma+1} \mathcal{M}_1^2 - \frac{\gamma-1}{\gamma+1} \\ \frac{T_2}{T_1} &= \frac{P_2 \rho_2}{P_1 \rho_1} = \frac{\gamma-1}{\gamma+1} \left[\frac{2}{\gamma+1} \left(\gamma \mathcal{M}_1^2 - \frac{1}{\mathcal{M}_1^2} \right) + \frac{4\gamma}{\gamma-1} - \frac{\gamma-1}{\gamma+1} \right] \end{aligned}$$

Here we have used that for an **ideal gas**

$$P = (\gamma - 1) \rho \varepsilon = \frac{k_B T}{\mu m_p} \rho$$

Given that $\mathcal{M}_1 > 1$, we see that $\rho_2 > \rho_1$ (shocks **compress**), $u_2 < u_1$ (shocks **decelerate**), $P_2 > P_1$ (shocks **increase pressure**), and $T_2 > T_1$ (shocks **heat**). The latter may seem surprising, given that the shock is considered to be **adiabatic**: although the process has been adiabatic, in that $dQ/dt = 0$, the gas **has** changed its adiabat; its entropy has increased as a consequence of the shock converting kinetic energy into thermal, internal energy. In general, in the presence of **viscosity**, a change that is adiabatic does not imply that the states before and after are simply linked by the relation $P = K \rho^\gamma$, with K some constant. Shocks are always viscous, which causes K to change across the shock, such that the entropy increases; it is this aspect of the shock that causes **irreversibility**, thus defining an "arrow of time".

Back to the RH jump conditions: in the limit $\mathcal{M}_1 \gg 1$ we have that

$$\rho_2 = \frac{\gamma + 1}{\gamma - 1} \rho_1 = 4 \rho_1$$

where we have used that $\gamma = 5/3$ for a monoatomic gas. Thus, with an **adiabatic shock** you can achieve a maximum compression in density of a factor four! Physically, the reason why there is a maximal compression is that the pressure and temperature of the downstream fluid diverge as $\mathcal{M}_1^2$. This huge increase in downstream pressure inhibits the amount of compression of the downstream gas. However, this is only true under the assumption that the shock is **adiabtic**. The downstream, post-shocked gas is out of thermal equilibrium, and in general will be cooling (i.e., $\mathcal{L} > 0$). At a certain distance past the shock (i.e., when $x = x_3$ in Fig. 14), the fluid will re-establish **thermal equilibrium** (i.e., $\mathcal{L} = 0$). In some special cases, one can obtain the properties of the fluid in the new equilibrium state; one such case is the example of an **isothermal shock**, for which the downstream gas has the same temperature as the upstream gas (i.e., $T_3 = T_1$).

In the case of an **isothermal shock**, the first two **Rankine-Hugoniot jump con-**

ditions are still valid, i.e.,

$$\begin{aligned}\rho_1 u_1 &= \rho_3 u_3 \\ \rho_1 u_1^2 + P_1 &= \rho_3 u_3^2 + P_3\end{aligned}$$

However, the third condition, which derives from the energy equation, is no longer valid. After all, in deriving that one we had assumed that the shock was adiabatic. In the case of an isothermal shock we have to replace the third RH jump condition with $T_1 = T_3$. The latter implies that $c_s^2 = P_3/\rho_3 = P_1/\rho_1$, and allows us to rewrite the second RH condition as

$$\begin{aligned}\rho_1(u_1^2 + c_s^2) &= \rho_3(u_3^2 + c_s^2) \\ \Leftrightarrow u_1^2 - \frac{\rho_3}{\rho_1}u_3^2 &= \frac{\rho_3}{\rho_1}c_s^2 - c_s^2 \\ \Leftrightarrow u_1^2 - u_1u_3 &= \left(\frac{u_1}{u_3} - 1\right)c_s^2 \\ \Leftrightarrow u_1u_3(u_1 - u_3) &= (u_1 - u_3)c_s^2 \\ \Leftrightarrow c_s^2 &= u_1u_3\end{aligned}$$

Here the second step follows from using the first RH jump condition. If we now substitute this result back into the first RH jump condition we obtain that

$$\frac{\rho_3}{\rho_1} = \frac{u_1}{u_3} = \left(\frac{u_1}{c_s}\right)^2 = \mathcal{M}_1^2$$

Hence, in the case of **isothermal shock** (or an adiabatic shock, but sufficiently far behind the shock in the downstream fluid), we have that there is no restriction to how much compression the shock can achieve; depending on the Mach number of the shock, the compression can be huge.



Figure 15: *An actual example of a supernova blastwave. The red colors show the optical light emitted by the supernova ejecta, while the green colors indicate X-ray emission coming from the hot bubble of gas that has been shock-heated when the blast-wave ran over it.*

Supernova Blastwave: An important example of a shock in astrophysics are supernova blastwaves. When a supernova explodes, it blasts a shell of matter (the ‘ejecta’) at high (highly supersonic) speed into the surrounding medium. The kinetic energy of this shell material is roughly $E_{\text{SN}} = 10^{51}$ erg. This is roughly 100 times larger than the amount of energy emitted in radiation by the supernova explosion (which is what we ‘see’). For comparison, the entire Milky Way has a luminosity of $\sim 10^{10} L_{\odot} \simeq 4 \times 10^{43} \text{ erg s}^{-1}$, which amounts to an energy emitted by stars over an entire year that is of the order of 1.5×10^{51} erg. Hence, the kinetic energy released by a single SN is larger than the energy radiated by stars, by the entire galaxy, in an entire year!

The mass of the ejecta is of the order of 1 Solar mass, which implies (using that $E_{\text{SN}} = \frac{1}{2} M_{\text{ej}} v_{\text{ej}}^2$), that the ejecta have a velocity of $\sim 10,000 \text{ km s}^{-1}$!! Initially, this shell material has a mass that is much larger than the mass of the surroundings swept up by the shock, and to lowest order the shell undergoes free expansion. This phase is therefore called the **free-expansion phase**. As the shock moves out, it engulves more and more interstellar material, which is heated (and compressed) by the shock. Hence, the interior of the shell (=shock) is a super-hot bubble of over-pressurized

gas, which ‘pushes’ the shock outwards. As more and more material is swept-up, and accelerated outwards, the mass of the shell increases, which causes the velocity of the shell to decelerate. At the early stages, the cooling of the hot bubble is negligible, and the blastwave is said to be in the **adiabatic phase**, also known as the **Sedov-Taylor phase**. At some point, though, the hot bubble starts to cool, radiating away the kinetic energy of the supernova, and lowering the interior pressure up to the point that it no longer pushes the shell outwards. This is called the **radiative phase**. From this point on, the shell expands purely by its inertia, being slowed down by the work it does against the surrounding material. This phase is called the **snow-plow phase**. Ultimately, the velocity of the shell becomes comparable to the sound speed of the surrounding material, after which it continues to move outward as a sound wave, slowly dissipating into the surroundings.

During the adiabatic phase, we can use a simple dimensional analysis to solve for the evolution of the shock radius, r_{sh} , with time. Since the only physical parameters that can determine r_{sh} in this adiabatic phase are time, t , the initial energy of the SN explosion, ε_0 , and the density of the surrounding medium, ρ_0 , we have that

$$r_{\text{sh}} = f(t, \varepsilon_0, \rho_0) = At^\eta \varepsilon_0^\alpha \rho_0^\beta$$

It is easy to check that there is only one set of values for η , α and β for which the product on the right has the dimensions of length (which is the dimension of r_{sh}). This solution has $\eta = 2/5$, $\alpha = 1/5$ and $\beta = -1/5$, such that

$$r_{\text{sh}} = A \left(\frac{\varepsilon}{\rho_0} \right)^{1/5} t^{2/5}$$

and thus

$$v_{\text{sh}} = \frac{dr_{\text{sh}}}{dt} = \frac{2A}{5} \left(\frac{\varepsilon}{\rho_0} \right)^{1/5} t^{-3/5}$$

which shows that indeed the shock decelerates as it moves outwards.

CHAPTER 13

Fluid Instabilities

In this Chapter we discuss the following instabilities:

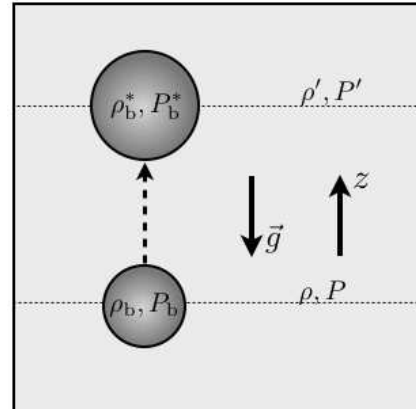
- convective instability (Schwarzschild criterion)
- interface instabilities (Rayleigh-Taylor & Kelvin-Helmholtz)
- gravitational instability (Jeans criterion)
- thermal instability (Field criterion)

Convective Instability: In astrophysics we often need to consider fluids heated from "below" (e.g., stars, Earth's atmosphere where Sun heats surface, etc.)³. This results in a temperature gradient: hot at the base, colder further "up". Since warmer fluids are more buoyant ('lighter'), they like to be further up than colder ('heavier') fluids. The question we need to address is under what conditions this adverse temperature gradient becomes unstable, developing "overturning" motions known as thermal **convection**.

Consider a blob with density ρ_b and pressure P_b embedded in an ambient medium of density ρ and pressure P . Suppose the blob is displaced by a small distance δz upward. After the displacement the blob will have conditions (ρ_b^*, P_b^*) and its new ambient medium is characterized by (ρ', P') , where

$$\rho' = \rho + \frac{d\rho}{dz} \delta z \quad P' = P + \frac{dP}{dz} \delta z$$

Initially the blob is assumed to be in **mechanical** and **thermal equilibrium** with its ambient medium, so that $\rho_b = \rho$ and $P_b = P$. After the displacement the blob needs to re-establish a new mechanical and thermal equilibrium. In general, the time scale on which it re-establishes mechanical (pressure) equilibrium is the sound crossing



³Here and in what follows, 'up' refers to the direction opposite to that of gravity.

time, τ_s , while re-establishing thermal equilibrium proceeds much slower, on the conduction time, τ_c . Given that $\tau_s \ll \tau_c$ we can assume that $P_b^* = P'$, and treat the displacement as **adiabatic**. The latter implies that the process can be described by an adiabatic EoS: $P \propto \rho^\gamma$. Hence, we have that

$$\rho_b^* = \rho_b \left(\frac{P_b^*}{P_b} \right)^{1/\gamma} = \rho_b \left(\frac{P'}{P} \right)^{1/\gamma} = \rho_b \left[1 + \frac{1}{P} \frac{dP}{dz} \delta z \right]^{1/\gamma}$$

In the limit of small displacements δz , we can use Taylor series expansion to show that, to first order,

$$\rho_b^* = \rho + \frac{\rho}{\gamma P} \frac{dP}{dz} \delta z$$

where we have used that initially $\rho_b = \rho$, and that the Taylor series expansion, $f(x) \simeq f(0) + f'(0)x + \frac{1}{2}f''(0)x^2 + \dots$, of $f(x) = [1+x]^{1/\gamma}$ is given by $f(x) \simeq 1 + \frac{1}{\gamma}x + \dots$. Suppose we have a stratified medium in which $d\rho/dz < 0$ and $dP/dz < 0$. In that case, if $\rho_b^* > \rho'$ the blob will be heavier than its surrounding and it will sink back to its original position; the system is stable to convection. If, on the other hand, $\rho_b^* < \rho'$ then the displacement has made the blob more buoyant, resulting in **instability**. Hence, using that $\rho' = \rho + (d\rho/dz) \delta z$ we see that stability requires that

$$\frac{d\rho}{dz} < \frac{\rho}{\gamma P} \frac{dP}{dz}$$

This is called the **Schwarzschild criterion for convective stability**.

It is often convenient to rewrite this criterion in a form that contains the temperature. Using that

$$\rho = \rho(P, T) = \frac{\mu m_p}{k_B T} P$$

it is straightforward to show that

$$\frac{d\rho}{dz} = \frac{\rho}{P} \frac{dP}{dz} - \frac{\rho}{T} \frac{dT}{dz}$$

Substitution in $\rho' = \rho + (d\rho/dz) \delta z$ then yields that

$$\rho_b^* - \rho' = \left[-\left(1 - \frac{1}{\gamma}\right) \frac{\rho}{P} \frac{dP}{dz} + \frac{\rho}{T} \frac{dT}{dz} \right] \delta z$$

Since stability requires that $\rho_b^* - \rho' > 0$, and using that $\delta z > 0$, $dP/dz < 0$ and $dT/dz < 0$ we can rewrite the above Schwarzschild criterion for stability as

$$\left| \frac{dT}{dz} \right| < \left(1 - \frac{1}{\gamma} \right) \frac{T}{P} \left| \frac{dP}{dz} \right|$$

This shows that if the temperature gradient becomes too large the system becomes convectively unstable: blobs will rise up until they start to loose their thermal energy to the ambient medium, resulting in convective energy transport that tries to “overturn” the hot (high entropy) and cold (low entropy) material. In fact, without any proof we mention that in terms of the **specific entropy**, s , one can also write the Schwarzschild criterion for convective stability as $ds/dz > 0$.

To summarize, the **Schwarzschild criterion for convective stability** is given by either of the following three expressions:

$$\boxed{\begin{aligned} \left| \frac{dT}{dz} \right| &< \left(1 - \frac{1}{\gamma} \right) \frac{T}{P} \left| \frac{dP}{dz} \right| \\ \frac{d\rho}{dz} &< \frac{\rho}{\gamma P} \frac{dP}{dz} \\ \frac{ds}{dz} &> 0 \end{aligned}}$$

Rayleigh-Taylor Instability: The Rayleigh-Taylor (RT) instability is an instability of an interface between two fluids of different densities that occurs when one of the fluids is accelerated into the other. Examples include supernova explosions in which expanding core gas is accelerated into denser shell gas and the common terrestrial example of a denser fluid such as water suspended above a lighter fluid such as oil in the Earth’s gravitational field.

It is easy to see where the RT instability comes from. Consider a fluid of density ρ_2 sitting on top of a fluid of density $\rho_1 < \rho_2$ in a gravitational field that is pointing in the downward direction. Consider a small perturbation in which the initially horizontal interface takes on a small amplitude, sinusoidal deformation. Since this

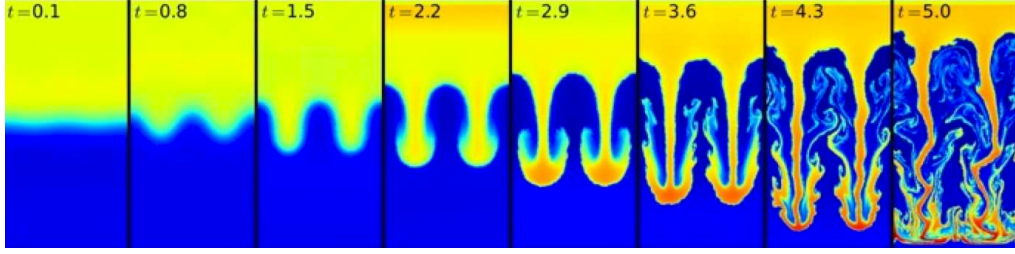


Figure 16: *Example of Rayleigh-Taylor instability in a hydro-dynamical simulation.*

implies moving a certain volume of denser material down, and an equally large volume of the lighter material up, it is immediately clear that the potential energy of this ‘perturbed’ configuration is lower than that of the initial state, and therefore energetically favorable. Simply put, the initial configuration is unstable to small deformations of the interface.

Stability analysis (i.e., perturbation analysis of the fluid equations) shows that the **dispersion relation** corresponding to the RT instability is given by

$$\omega = \pm i k \sqrt{\frac{g}{k} \frac{\rho_2 - \rho_1}{\rho_2 + \rho_1}}$$

where g is the gravitational acceleration, and the factor $(\rho_2 - \rho_1)/(\rho_2 + \rho_1)$ is called the **Atwood number**. Since the wavenumber of the perturbation $k > 0$ we see that ω is imaginary, which implies that the perturbations will grow exponentially (i.e., the system is unstable). If $\rho_1 > \rho_2$ though, ω is real, and the system is stable (perturbations to the interface propagate as waves).

Kelvin-Helmholtz Instability: the Kelvin-Helmholtz (KH) instability is an interface instability that arises when two fluids with different densities have a velocity difference across their interface. Similar to the RT instability, the KH instability manifests itself as a small wavy pattern in the interface which develops into turbulence and which causes **mixing**. Examples where KH instability plays a role are wind blowing over water, (astrophysical) jets, the cloud bands on Jupiter (in particular the famous red spot), and clouds of denser gas falling through the hot, low density intra-cluster medium (ICM).

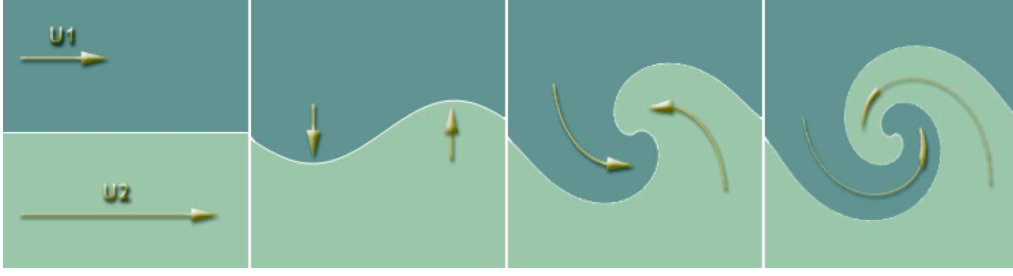


Figure 17: *Illustration of onset of Kelvin-Helmholtz instability*

Stability analysis (i.e., perturbation analysis of the fluid equations) shows that the the **dispersion relation** corresponding to the KH instability is given by

$$\frac{\omega}{k} = \frac{(\rho_1 u_1 + \rho_2 u_2) \pm i (u_1 - u_2) (\rho_1 \rho_2)^{1/2}}{\rho_1 + \rho_2}$$

Note that this dispersion relation has both real and imaginary parts, given by

$$\frac{\omega_R}{k} = \frac{(\rho_1 u_1 + \rho_2 u_2)}{\rho_1 + \rho_2}$$

and

$$\frac{\omega_I}{k} = (u_1 - u_2) \frac{(\rho_1 \rho_2)^{1/2}}{\rho_1 + \rho_2}$$

Since the imaginary part is non-zero, except for $u_1 = u_2$, we we have that, in principle, *any* velocity difference across an interface is KH unstable. In practice, **surface tension** can stabilize the short wavelength modes so that typically **KH instability** kicks in above some velocity treshold.

As an example, consider a cold cloud of radius R_c falling into a cluster of galaxies. The latter contains a hot intra-cluster medium (ICM), and as the cloud moves through this hot ICM, KH instabilities can develop on its surface. If the cloud started out at a large distance from the cluster with zero velocity, than at infall it has a velocity $v \sim v_{\text{esc}} \sim c_{\text{s,h}}$, where the latter is the sound speed of the hot ICM, assumed to be in hydrostatic equilibrium. Defining the cloud's overdensity $\delta = \rho_c/\rho_h - 1$, we can write the (imaginary part of the) dispersion relation as

$$\omega = \frac{\rho_h (\rho_c/\rho_h)^{1/2}}{\rho_h [1 + (\rho_c/\rho_h)]} c_{\text{s,h}} k = \frac{(\delta + 1)^{1/2}}{\delta + 2} c_{\text{s,h}} k$$

The mode that will destroy the cloud has $k \sim 1/R_c$, so that the time-scale for cloud destruction is

$$\tau_{\text{KH}} \simeq \frac{1}{\omega} \simeq \frac{R_c}{c_{\text{s,h}}} \frac{\delta + 2}{(\delta + 1)^{1/2}}$$

Assuming pressure equilibrium between cloud and ICM, and adopting the EoS of an ideal gas, implies that $\rho_h T_h = \rho_c T_c$, so that

$$\frac{c_{\text{s,h}}}{c_{\text{s,c}}} = \frac{T_h^{1/2}}{T_c^{1/2}} = \frac{\rho_c^{1/2}}{\rho_h^{1/2}} = (\delta + 1)^{1/2}$$

Hence, one finds that the **Kelvin-Helmholtz time** for cloud destruction is

$$\tau_{\text{KH}} \simeq \frac{1}{\omega} \simeq \frac{R_c}{c_{\text{s,c}}} \frac{\delta + 2}{\delta + 1}$$

Note that $\tau_{\text{KH}} \sim \zeta(R_c/c_{\text{s,c}}) = \zeta\tau_s$, with $\zeta = 1(2)$ for $\delta \gg 1(\ll 1)$. Hence, the **Kelvin-Helmholtz instability** will typically destroy clouds falling into a hot "atmosphere" on a time scale between one and two **sound crossing times**, τ_s , of the cloud. Note, though, that magnetic fields and/or radiative cooling at the interface may stabilize the clouds.

Gravitational Instability: In our discussion of sound waves we used perturbation analysis to derive a dispersion relation $\omega^2 = k^2 c_s^2$. In deriving that equation we ignored gravity by setting $\nabla\Phi = 0$ (see Chapter 13). If you do not ignore gravity, then you add one more perturbed quantity; $\Phi = \Phi_0 + \Phi_1$ and one more equation, namely the **Poisson equation** $\nabla^2\Phi = 4\pi G\rho$.

It is not difficult to show that this results in a modified **dispersion relation**:

$$\boxed{\omega^2 = k^2 c_s^2 - 4\pi G\rho_0 = c_s^2 (k^2 - k_J^2)}$$

where we have introduced the **Jeans wavenumber**

$$k_J = \frac{\sqrt{4\pi G\rho_0}}{c_s}$$

to which we can also associate a **Jeans length**

$$\boxed{\lambda_J \equiv \frac{2\pi}{k_J} = \sqrt{\frac{\pi}{G\rho_0}} c_s}$$

and a **Jeans mass**

$$M_J = \frac{4}{3}\pi\rho_0 \left(\frac{\lambda_J}{2}\right)^3 = \frac{\pi}{6}\rho_0 \lambda_J^3$$

From the dispersion relation one immediately sees that the system is **unstable** (i.e., ω is imaginary) if $k < k_J$ (or, equivalently, $\lambda > \lambda_J$ or $M > M_J$). This is called the **Jeans criterion for gravitational instability**. It expresses when pressure forces (which try to disperse matter) are no longer able to overcome gravity (which tries to make matter collapse), resulting in exponential gravitational collapse on a time scale

$$\tau_{\text{ff}} = \sqrt{\frac{3\pi}{32G\rho}}$$

known as the **free-fall time** for gravitational collapse.

The Jeans stability criterion is of utmost importance in astrophysics. It is used to describes the formation of galaxies and large scale structure in an expanding space-time (in this case the growth-rate is not exponential, but only power-law), to describe the formation of stars in molecular clouds within galaxies, and it may even play an important role in the formation of planets in protoplanetary disks.

In deriving the Jeans Stability criterion you will encounter a somewhat puzzling issue. Consider the **Poisson equation** for the unperturbed medium (which has density ρ_0 and gravitational potential Φ_0):

$$\nabla^2\Phi_0 = 4\pi G\rho_0$$

Since the initial, unperturbed medium is supposed to be homogeneous there can be no gravitational force; hence $\nabla\Phi_0 = 0$ everywhere. The above Poisson equation then implies that $\rho_0 = 0$. In other words, an unperturbed, homogeneous density field of non-zero density does not seem to exist. Sir James Jeans ‘ignored’ this ‘nuisance’ in his derivation, which has since become known as the **Jeans swindle**. The problem arises because Newtonian physics is not equipped to deal with systems of infinite extent (a requirement for a perfectly homogeneous density distribution). See Kiessling (1999; arXiv:9910247) for a detailed discussion, including an elegant demonstration that the Jeans swindle is actually vindicated!

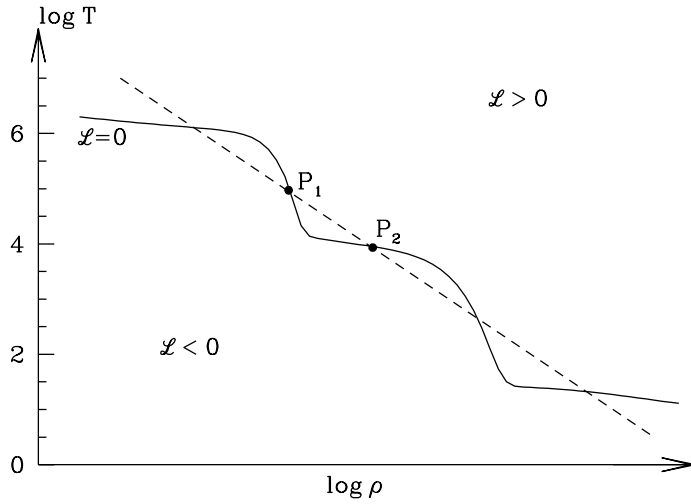


Figure 18: *The locus of thermal equilibrium ($\mathcal{L} = 0$) in the (ρ, T) plane, illustrating the principle of thermal instability. The dashed line indicates a line of constant pressure.*

Thermal Instability: Let $\mathcal{L} = \mathcal{L}(\rho, T) = \mathcal{C} - \mathcal{H}$ be the net cooling rate. If $\mathcal{L} = 0$ the system is said to be in **thermal equilibrium** (TE), while $\mathcal{L} > 0$ and $\mathcal{L} < 0$ correspond to cooling and heating, respectively.

The condition $\mathcal{L}(\rho, T) = 0$ corresponds to a curve in the (ρ, T) -plane with a shape similar to that shown in Fig. 18. It has flat parts at $T \sim 10^6$ K, at $T \sim 10^4$ K, at $T \sim 10 - 100$ K. This can be understood from simple atomic physics (see for example § 8.5.1 of Mo, van den Bosch & White, 2010). Above the TE curve we have that $\mathcal{L} > 0$ (net cooling), while below it $\mathcal{L} < 0$ (net heating). The dotted curve indicates a line of constant pressure ($T \propto \rho^{-1}$). Consider a blob in thermal and mechanical (pressure) equilibrium with its ambient medium, and with a pressure indicated by the dashed line. There are five possible solutions for the density and temperature of the blob, two of which are indicated by P_1 and P_2 ; here confusingly the P refers to ‘point’ rather than ‘pressure’. Suppose I have a blob located at point P_2 . If I heat the blob, displacing it from TE along the constant pressure curve (i.e., the blob is assumed small enough that the sound crossing time, on which the blob re-established mechanical equilibrium, is short). The blob now finds itself in the region where $\mathcal{L} > 0$ (i.e, net cooling), so that it will cool back to its original location on the TE-curve; the blob is **stable**. For similar reasons, it is easy to see that a blob located at point P_1 is **unstable**. This instability is called **thermal instability**, and it explains why the ISM is a **three-phase** medium, with gas of three different temperatures ($T \sim 10^6$ K, 10^4 K, and $\sim 10 - 100$ K) coexisting in pressure equilibrium. Gas at any other temperature but in pressure equilibrium is thermally unstable.

It is easy to see that the requirement for **thermal instability** translates into

$$\left(\frac{\partial \mathcal{L}}{\partial T}\right)_P < 0$$

which is known as the **Field criterion for thermal instability** (after astrophysicist George B. Field).

Fragmentation and Shattering: Consider the Jeans criterion, expressing a balance between gravity and pressure. Using that the **Jeans mass** $M_J \propto \rho \lambda_J^3$ and that $\lambda_J \propto \rho^{-1/2} c_s$, we see that

$$M_J \propto \rho^{-1/2} T^{3/2}$$

where we have used that $c_s \propto T^{1/2}$. Now consider a **polytropic equation of state**, which has $P \propto \rho^\Gamma$, with Γ the **polytropic index**. Assuming an ideal gas, such that

$$P = \frac{k_B T}{\mu m_p} \rho$$

we thus see that a polytropic ideal gas must have that $T \propto \rho^{\Gamma-1}$. Substituting that in the expression for the Jeans mass, we obtain that

$$M_J \propto \rho^{\frac{3}{2}\Gamma-2} = \rho^{\frac{3}{2}(\Gamma-\frac{4}{3})}$$

Thus, we see that for $\Gamma > 4/3$ the Jeans mass will increase with increasing density, while the opposite is true for $\Gamma < 4/3$. Now consider a system that is (initially) larger than the Jeans mass. Since pressure can no longer support it against its own gravity, the system will start to collapse, which increases the density. If $\Gamma < 4/3$, the Jeans mass will become smaller as a consequence of the collapse, and now small subregions of the system will find themselves having a mass larger than the Jeans mass $\Rightarrow$ the system will start to **fragment**.

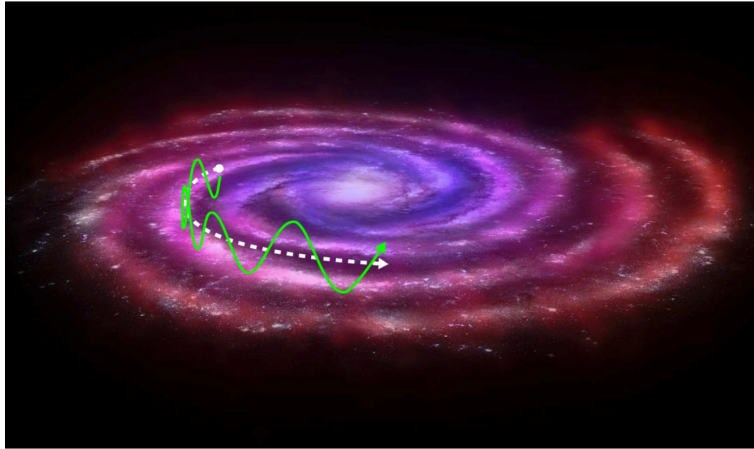
If the collapse is adiabatic (i.e., we can ignore cooling), then $\Gamma = \gamma = 5/3 > 4/3$ and there will be no fragmentation. However, if cooling is very efficient, such that while the cloud collapses it maintains the same temperature, the EoS is now isothermal, which implies that $\Gamma = 1 < 4/3$: the cloud will fragment into smaller collapsing clouds. Fragmentation is believed to underly the formation of star clusters.

A very similar process operates related to the **thermal instability**. In the discussion of the **Field criterion** we had made the assumption “*the blob is assumed small enough that the sound crossing time, on which the blob re-established mechanical equilibrium, is short*”. Here ‘short’ means compared to the cooling time of the cloud. Let’s define the **cooling length** $l_{\text{cool}} \equiv c_s \tau_{\text{cool}}$, where c_s is the cloud’s sound speed and τ_{cool} is the cooling time (the time scale on which it radiates away most of its internal energy). The above assumption thus implies that the size of the cloud, $l_{\text{cloud}} \ll l_{\text{cool}}$. As a consequence, whenever the cloud cools somewhat, it can immediately re-establish pressure equilibrium with its surrounding (i.e., the sound crossing time, $\tau_s = l_{\text{cloud}}/c_s$ is much smaller than the cooling time $\tau_{\text{cool}} = l_{\text{cool}}/c_s$).

Now consider a case in which $l_{\text{cloud}} \gg l_{\text{cool}}$ (i.e., $\tau_{\text{cool}} \ll \tau_s$). As the cloud cools, it cannot maintain pressure equilibrium with its surroundings; it takes too long for mechanical equilibrium to be established over the entire cloud. What happens is that smaller subregions, of order the size l_{cool} , will fragment. The smaller fragments will be able to maintain pressure equilibrium with their surroundings. But as the small cloudlets cool further, the cooling length l_{cool} shrinks. To see this, realize that when T drops this lowers the sound speed and decreases the cooling time; after all, we are in the regime of thermal instability, so $(\partial \mathcal{L} / \partial T)_P < 0$. As a consequence, $l_{\text{cool}} = c_s \tau_{\text{cool}}$ drops as well. So the small cloudlett soon finds itself larger than the cooling length, and it in turn will fragment. This process of **shattering** continues until the cooling time becomes sufficiently long and the cloudletts are no longer thermally unstable (see McCourt et al., 2018, MNRAS, **473**, 5407 for details).

This process of **shattering** is believed to play an important role in the inter-galactic medium (IGM) in between galaxies, and the circum-galactic medium (CGM) in the halos of galaxies.

Part II: Collisionless Dynamics



The following chapters give an elementary introduction into the rich topic of collisionless dynamics. The main goal is to highlight how the lack of collisions among the constituent particles give rise to a dynamics that differs remarkably from collisional fluids. We also briefly discuss the theory of orbits, which are the building blocks of collisionless systems, the Virial theorem, and the gravothermal catastrophe, which is a consequence of the negative heat capacity of a gravitational system. Finally, we briefly discuss interactions (‘collisions’) among collisionless systems.

Collisionless Dynamics is a rich topic, and one could easily devote an entire course to it (for example the Yale Graduate Course ‘ASTR 518; Galactic Dynamics’). The following chapters therefore only scratch the surface of this rich topic. Readers who want to get more indepth information are referred to the following excellent textbooks

- *Galactic Dynamics* by J. Binney & S. Tremaine
- *Galactic Nuclei* by D. Merritt
- *Galaxy Formation and Evolution* by H.J. Mo, F. van den Bosch & S. White

CHAPTER 14

Potential Theory & The Virial Theorem

Gravity in Astrophysical Fluids: Many of the fluids encountered in astrophysics are self-gravitating, which means that the gravitational force due to the fluid itself exceeds the gravitational force from the external mass distribution. Arguably the most important example of self-gravitating, astrophysical fluids are stars. But Cold Dark Matter halos are also examples of self-gravitating fluids (albeit collisionless). The interstellar medium (ISM) can and cannot be self-gravitating, depending on the conditions. The intra-cluster medium (ICM) is generally not self-gravitating; rather the gravitating potential is dominated by the dark matter.

Gravitational Potential: Gravity is a conservative force, which means that it can be written as the gradient of a scalar field. Newton's gravitational potential, $\Phi(\vec{x})$, is defined such that the gravitational force per unit mass

$$\vec{F}_g = -\nabla\Phi$$

Note that the absolute normalization of Φ has no physical relevance; only the gradients of Φ matter.

Consider a density distribution $\rho(\vec{x})$. What is the gravitational force $\vec{F}_g$ acting on a particle of mass m at location $\vec{x}$? We can sum the small contributions $\delta\vec{F}_g$ from different regions $\vec{x}' \pm d^3\vec{x}'$, given by

$$\delta\vec{F}_g(\vec{x}) = G \frac{m \delta m(\vec{x}')}{|\vec{x}' - \vec{x}|^2} \frac{\vec{x}' - \vec{x}}{|\vec{x}' - \vec{x}|} = Gm \frac{\vec{x}' - \vec{x}}{|\vec{x}' - \vec{x}|^3} \rho(\vec{x}') d^3\vec{x}'$$

Adding up all the small contributions yields $\vec{F}_g(\vec{x}) = \int \delta\vec{F}_g(\vec{x}) \equiv m \vec{g}(\vec{x})$, where

$$\vec{g}(\vec{x}) = G \int d^3\vec{x}' \frac{\vec{x}' - \vec{x}}{|\vec{x}' - \vec{x}|^3} \rho(\vec{x}')$$

is the **gravitational field** (i.e., the force per unit mass). Using that

$$\frac{\vec{x}' - \vec{x}}{|\vec{x}' - \vec{x}|^3} = \nabla_x \left(\frac{1}{|\vec{x}' - \vec{x}|} \right)$$

we can rewrite $g(\vec{x})$ as

$$\vec{g}(\vec{x}) = G \int d^3\vec{x}' \nabla_x \left(\frac{1}{|\vec{x}' - \vec{x}|} \right) \rho(\vec{x}') = \nabla_x \int d^3\vec{x}' \frac{G\rho(\vec{x}')}{|\vec{x}' - \vec{x}|} \equiv -\nabla_x \Phi$$

where in the last step we have defined the gravitational potential

$$\Phi(\vec{x}) = -G \int d^3\vec{x}' \frac{\rho(\vec{x}')}{|\vec{x}' - \vec{x}|}$$

It can be shown, that the above expression is equivalent to what is known as the **Poisson equation**:

$$\nabla^2 \Phi = 4\pi G \rho$$

For a derivation, see Section 3.2 of **Astrophysical Fluid Dynamics** by Clarke & Carswell, or Section 2.1 of **Galactic Dynamics** by Binney & Tremaine.

In general, it is extremely complicated to solve the **Poisson equation** for $\Phi(\vec{x})$ given $\rho(\vec{x})$ [see Chapter 2 of **Galactic Dynamics** by Binney & Tremaine for a detailed discussion]. However, under certain symmetries, solutions to the Poisson equation are fairly straightforward. In particular, under **spherical symmetry** the general solution to the **Poisson equation** is

$$\Phi(r) = -4\pi G \left[\frac{1}{r} \int_0^r \rho(r') r'^2 dr' + \int_r^\infty \rho(r') r' dr' \right]$$

Note that the potential at r depends on the mass distribution outside of r . However, if we now compute the gravitational force per unit mass

$$\vec{F}_g(r) = -\frac{d\Phi}{dr} \hat{e}_r = -\frac{G M(r)}{r^2} \hat{e}_r$$

where

$$M(r) \equiv 4\pi \int_0^r \rho(r') r'^2 dr$$

is the **enclosed mass** within r . This shows that the gravitational force does *not* depend on the mass distribution outside of r .

Newton's first theorem: a body that is inside a spherical shell of matter experiences no net gravitational force from that shell. The equivalent in general relativity is called **Birkhoff's theorem**.

This is easily understood from the fact that the solid angles that extent from a point inside a sphere to opposing directions have areas on the sphere that scale as r^2 (where r is the distance from the point to the sphere), while the gravitational force per unit mass scales as r^{-2} . Hence, the gravitational forces from the two opposing areas exactly cancel.

Circular velocity: the velocity of a particle or fluid element on a circular orbit. For a spherical mass distribution

$$V_{\text{circ}}(r) = \sqrt{r \frac{d\Phi}{dr}} = \sqrt{\frac{G M(r)}{r}}$$

In the case of an axisymmetric mass distribution, the circular velocity in the **equatorial plane** ($z = 0$, where z is one of the three cylindrical coordinates (R, ϕ, z)) is given by

$$V_{\text{circ}}(R) = \sqrt{R \frac{d\Phi}{dR}} \neq \sqrt{\frac{G M(R)}{R}}$$

Escape velocity: the velocity needed for a particle or fluid element to escape to infinity. Since $E = v^2/2 + \Phi(\vec{x})$, and escape requires $E > 0$, the escape velocity is

$$V_{\text{esc}}(\vec{x}) = \sqrt{2 |\Phi(\vec{x})|}$$

independent of the symmetry (or lack thereof) of the mass distribution.

Since gas cannot be on **self-intersecting orbits**, gas in disk galaxies generally orbits on circular orbits. The measured rotation velocities therefore reflect the circular velocities, which can be used to infer the enclosed mass as a function of radius. This method is generally used to infer the presence of **dark matter halos** surrounding disk galaxies.

Consider a gravitational system consisting of N particles (e.g., stars, fluid elements). The **total energy** of the system is $E = K + W$, where

$$\begin{aligned} \textbf{Total Kinetic Energy:} \quad K &= \sum_{i=1}^N \frac{1}{2} m_i v_i^2 \\ \textbf{Total Potential Energy:} \quad W &= -\frac{1}{2} \sum_{i=1}^N \sum_{j \neq i}^N \frac{G m_i m_j}{|\vec{r}_i - \vec{r}_j|} \end{aligned}$$

The latter follows from the fact that **gravitational binding energy** between a pair of masses is proportional to the product of their masses, and inversely proportional to their separation. The factor $1/2$ corrects for double counting the number of pairs.

Potential Energy in Continuum Limit: To infer an expression for the gravitational potential energy in the continuum limit, it is useful to rewrite the above expression as

$$W = \frac{1}{2} \sum_{i=1}^N m_i \Phi_i$$

where

$$\Phi_i = - \sum_{j \neq i} \frac{G m_j}{r_{ij}}$$

where $r_{ij} = |\vec{r}_i - \vec{r}_j|$. In the continuum limit this simply becomes

$$W = \frac{1}{2} \int \rho(\vec{x}) \Phi(\vec{x}) d^3\vec{x}$$

One can show (see e.g., Binney & Tremaine 2008) that this is equal to the trace of the **Chandrasekhar Potential Energy Tensor**

$$W_{ij} \equiv - \int \rho(\vec{x}) x_i \frac{\partial \Phi}{\partial x_j} d^3\vec{x}$$

In particular,

$$W = \text{Tr}(W_{ij}) = \sum_{i=1}^3 W_{ii} = - \int \rho(\vec{x}) \vec{x} \cdot \nabla \Phi d^3\vec{x}$$

which is another, equally valid, expression for the gravitational potential energy in the continuum limit.

Virial Theorem: A stationary, gravitational system obeys

$$2K + W = 0$$

Actually, the correct virial equation is $2K + W + \Sigma = 0$, where Σ is the surface pressure. In many, but certainly not all, applications in astrophysics this term can be ignored. Many textbooks don't even mention the surface pressure term.

Combining the **virial equation** with the expression for the total energy, $E = K + W$, we see that for a system that obeys the virial theorem

$$E = -K = W/2$$

Example: Consider a cluster consisting of N galaxies. If the cluster is in virial equilibrium then

$$2 \sum_{i=1}^N \frac{1}{2} m v_i^2 - \frac{1}{2} \sum_{i=1}^N \sum_{j \neq i} \frac{G m_i m_j}{r_{ij}} = 0$$

If we assume, for simplicity, that all galaxies have equal mass then we can rewrite this as

$$N m \frac{1}{N} \sum_{i=1}^N v_i^2 - \frac{G (Nm)^2}{2} \frac{1}{N^2} \sum_{i=1}^N \sum_{j \neq i} \frac{1}{r_{ij}} = 0$$

Using that $M = Nm$ and $N(N-1) \simeq N^2$ for large N , this yields

$$M = \frac{2 \langle v^2 \rangle}{G \langle 1/r \rangle}$$

with

$$\langle 1/r \rangle = \frac{1}{N(N-1)} \sum_{i=1}^N \sum_{j \neq i} \frac{1}{r_{ij}}$$

It is useful to define the **gravitational radius** r_g such that

$$W = -\frac{G M^2}{r_g}$$

Using the relations above, it is clear that $r_g = 2/\langle 1/r \rangle$. We can now rewrite the above equation for M in the form

$$M = \frac{r_g \langle v^2 \rangle}{G}$$

Hence, one can infer the mass of our cluster of galaxies from its velocity dispersion and its gravitation radius. In general, though, neither of these is observable, and one uses instead

$$M = \alpha \frac{R_{\text{eff}} \langle v_{\text{los}}^2 \rangle}{G}$$

where v_{los} is the line-of-sight velocity, R_{eff} is some measure for the ‘effective’ radius of the system in question, and α is a parameter of order unity that depends on the radial distribution of the galaxies. Note that, under the assumption of **isotropy**, $\langle v_{\text{los}}^2 \rangle = \langle v^2 \rangle / 3$ and one can also infer the mean reciprocal pair separation from the projected pair separations; in other words under the assumption of isotropy one can infer α , and thus use the above equation to compute the total, gravitational mass of the cluster. This method was applied by **Fritz Zwicky** in 1933, who inferred that the total dynamical mass in the Coma cluster is much larger than the sum of the masses of its galaxies. This was the first observational evidence for **dark matter**, although it took the astronomical community until the late 70’s to generally accept this notion.

For a self-gravitating fluid

$$K = \sum_{i=1}^N \frac{1}{2} m_i v_i^2 = \frac{1}{2} N m \langle v^2 \rangle = \frac{3}{2} N k_B T$$

where the last step follows from the kinetic theory of ideal gases of monoatomic particles. In fact, we can use the above equation for *any* fluid (including a collisionless one), if we interpret T as an **effective** temperature that measures the rms velocity of the constituent particles. If the system is in virial equilibrium, then

$$E = -K = -\frac{3}{2} N k_B T$$

which, as we show next, has some important implications...

Heat Capacity: the amount of heat required to increase the temperature by one degree Kelvin (or Celsius). For a self-gravitating fluid this is

$$C \equiv \frac{dE}{dT} = -\frac{3}{2} N k_B$$

which is negative! This implies that **by losing energy, a gravitational system gets hotter!!** This is a very counter-intuitive result, that often leads to confusion and wrong expectations. Below we give three examples of implications of the **negative heat capacity** of gravitating systems,

Example 1: Drag on satellites Consider a satellite orbiting Earth. When it experiences friction against the (outer) atmosphere, it loses energy. This causes the system to become more strongly bound, and the orbital radius to shrink. Consequently, the energy loss results in the gravitational potential energy, W , becoming more negative. In order for the satellite to re-establish virial equilibrium ($2K + W = 0$), its kinetic energy needs to **increase**. Hence, contrary to common intuition, *friction causes the satellite to speed up*, as it moves to a lower orbit (where the circular velocity is higher).

Example 2: Stellar Evolution A star is a gaseous, self-gravitating sphere that radiates energy from its surface at a luminosity L . Unless this energy is replenished (i.e., via some energy production mechanism in the star's interior), the star will react by shrinking (i.e., the energy loss implies an increase in binding energy, and thus a

potential energy that becomes more negative). In order for the star to remain in virial equilibrium its kinetic energy, which is proportional to temperature, has to increase; *the star's energy loss results in an increase of its temperature.*

In the Sun, hydrogen burning produces energy that replenishes the energy loss from the surface. As a consequence, the system is in equilibrium, and will not contract. However, once the Sun has used up all its hydrogen, it will start to contract and heat up, because of the **negative heat capacity**. This continues until the temperature in the core becomes sufficiently high that helium can start to fuse into heavier elements, and the Sun settles in a new equilibrium.

Example 3: Core Collapse *a system with negative heat capacity in contact with a heat bath is thermodynamically unstable.* Consider a self-gravitating fluid of ‘temperature’ T_1 , which is in contact with a heat bath of temperature T_2 . Suppose the system is in thermal equilibrium, so that $T_1 = T_2$. If, due to some small disturbance, a small amount of heat is transferred from the system to the heat bath, the negative heat capacity implies that this results in $T_1 > T_2$. Since heat always flows from hot to cold, more heat will now flow from the system to the heat bath, further increasing the temperature difference, and T_1 will continue to rise without limit. This run-away instability is called the **gravothermal catastrophe**. An example of this instability is the **core collapse of globular clusters**: Suppose the formation of a gravitational system results in the system having a declining velocity dispersion profile, $\sigma^2(r)$ (i.e., σ decreases with increasing radius). This implies that the central region is (dynamically) hotter than the outskirts. **IF** heat can flow from the center to those outskirts, the gravothermal catastrophe kicks in, and σ in the central regions will grow without limits. Since $\sigma^2 = GM(r)/r$, the central mass therefore gets compressed into a smaller and smaller region, while the outer regions expand. This is called **core collapse**. Note that this does NOT lead to the formation of a supermassive black hole, because regions at smaller r always shrink faster than regions at somewhat larger r . In dark matter halos, and elliptical galaxies, the velocity dispersion profile is often declining with radius. However, in those systems the two-body relaxation time is so long that there is basically no heat flow (which requires two-body interactions). However, globular clusters, which consist of $N \sim 10^4$ stars, and have a crossing time of only $t_{\text{cross}} \sim 5 \times 10^6 \text{yr}$, have a two-body relaxation time of only $\sim 5 \times 10^8 \text{yr}$. Hence, heat flow in globular clusters is not negligible, and they can (and do) undergo core collapse. The collapse does not proceed indefinitely, because of binaries (see **Galactic Dynamics** by Binney & Tremaine for more details).

CHAPTER 15

Collisionless Dynamics: CBE & Jeans Equations

In this chapter we consider collisionless fluids, such as galaxies and dark matter halos. As discussed in previous chapters, their dynamics is governed by the **Collisionless Boltzmann equation** (CBE)

$$\frac{df}{dt} = \frac{\partial f}{\partial t} + v_i \frac{\partial f}{\partial x_i} - \frac{\partial \Phi}{\partial x_i} \frac{\partial f}{\partial v_i} = 0$$

By taking the velocity moment of the CBE (see Chapter 8), we obtain the **Jeans equations**

$$\frac{\partial u_i}{\partial t} + u_j \frac{\partial u_i}{\partial x_j} = -\frac{1}{\rho} \frac{\partial \hat{\sigma}_{ij}}{\partial x_j} - \frac{\partial \Phi}{\partial x_i}$$

which are the equivalent of the Navier-Stokes equations (or Euler equations), but for a collisionless fluid. The quantity $\hat{\sigma}_{ij}$ in the above expression is the **stress tensor**, defined as

$$\hat{\sigma}_{ij} = -\rho \langle w_i w_j \rangle = -\rho (\langle v_i v_j \rangle - \langle v_i \rangle \langle v_j \rangle)$$

In this chapter, we write a hat on top of the stress tensor, in order to distinguish it from the **velocity dispersion tensor** given by

$$\sigma_{ij}^2 = \langle v_i v_j \rangle - \langle v_i \rangle \langle v_j \rangle = -\frac{\hat{\sigma}_{ij}}{\rho}$$

This notation may cause some confusion, but it is adapted here in order to be consistent with the notation in standard textbooks on galactic dynamics. For the same reason, in what follows we will write $\langle v_i \rangle$ in stead of u_i (also because u_i was defined as the velocity of a fluid element, but for a collisionless fluid the concept of a fluid element is not defined).

As we have discussed in detail in Chapters 4 and 5, for a collisional fluid the stress tensor is given by

$$\hat{\sigma}_{ij} = -\rho \sigma_{ij}^2 = -P \delta_{ij} + \tau_{ij}$$

and therefore completely specified by two scalar quantities; the pressure P and the shear viscosity μ (as always, we ignore bulk viscosity). Both P and μ are related to ρ and T via **constitutive equations**, which allow for closure in the equations.

In the case of a collisionless fluid, though, no constitutive relations exist, and the (symmetric) velocity dispersion tensor has 6 unknowns. As a consequence, the Jeans equations do not form a closed set. Adding higher-order moment equations of the CBE will yield more equations, but this also adds new, higher-order unknowns such as $\langle v_i v_j v_k \rangle$, etc. As a consequence, the set of CBE moment equations never closes!

Note that σ_{ij}^2 is a **local** quantity; $\sigma_{ij}^2 = \sigma_{ij}^2(\vec{x})$. At each point $\vec{x}$ it defines the **velocity ellipsoid**; an ellipsoid whose principal axes are defined by the orthogonal eigenvectors of σ_{ij}^2 with lengths that are proportional to the square roots of the respective eigenvalues.

Since these eigenvalues are typically not the same, a collisionless fluid experiences **anisotropic** pressure-like forces. In order to be able to close the set of Jeans equations, it is common to make certain assumptions about the symmetry of the fluid. For example, a common assumption is that the fluid is **isotropic**, such that the (local) velocity dispersion tensor is specified by a single quantity; the local velocity dispersion σ^2 . Note, though, that if with this approach, a solution is found, the solution may not correspond to a physical distribution function (DF) (i.e., in order to be physical, $f \geq 0$ everywhere). Thus, although any real DF obeys the Jeans equations, not every solution to the Jeans equations corresponds to a physical DF!!!

As a worked out example, we now derive the Jeans equations under **cylindrical symmetry**. We therefore write the Jeans equations in the cylindrical coordinate system (R, ϕ, z) . The first step is to write the CBE in cylindrical coordinates.

$$\frac{df}{dt} = \frac{\partial f}{\partial t} + \dot{R} \frac{\partial f}{\partial R} + \dot{\phi} \frac{\partial f}{\partial \phi} + \dot{z} \frac{\partial f}{\partial z} + \dot{v}_R \frac{\partial f}{\partial v_R} + \dot{v}_\phi \frac{\partial f}{\partial v_\phi} + \dot{v}_z \frac{\partial f}{\partial v_z}$$

Recall from vector calculus (see Appendices A and D) that

$$\vec{v} = \dot{R} \vec{e}_R + R \dot{\phi} \vec{e}_\phi + \dot{z} \vec{e}_z = v_R \vec{e}_R + v_\phi \vec{e}_\phi + v_z \vec{e}_z$$

from which we obtain the acceleration vector

$$\vec{a} = \frac{d\vec{v}}{dt} = \ddot{R} \vec{e}_R + \dot{R} \dot{\vec{e}}_R + \dot{R} \dot{\phi} \vec{e}_\phi + R \ddot{\phi} \vec{e}_\phi + R \dot{\phi} \dot{\vec{e}}_\phi + \ddot{z} \vec{e}_z + \dot{z} \dot{\vec{e}}_z$$

Using that $\dot{\vec{e}}_R = \dot{\phi}\vec{e}_\phi$, $\dot{\vec{e}}_\phi = -\dot{\phi}\vec{e}_R$, and $\dot{\vec{e}}_z = 0$ we have that

$$\vec{a} = \left[\ddot{R} - R\dot{\phi}^2 \right] \vec{e}_R + \left[2\dot{R}\dot{\phi} + R\ddot{\phi} \right] \vec{e}_\phi + \ddot{z}\vec{e}_z$$

Next we use that

$$\begin{aligned} v_R = \dot{R} &\Rightarrow \dot{v}_R = \ddot{R} \\ v_\phi = R\dot{\phi} &\Rightarrow \dot{v}_\phi = \dot{R}\dot{\phi} + R\ddot{\phi} \\ v_z = \dot{z} &\Rightarrow \dot{v}_z = \ddot{z} \end{aligned}$$

to write the acceleration vector as

$$\vec{a} = \left[\dot{v}_R - \frac{v_\phi^2}{R} \right] \vec{e}_R + \left[\frac{v_R v_\phi}{R} + \dot{v}_\phi \right] \vec{e}_\phi + \dot{v}_z \vec{e}_z$$

Newton's equation of motion in vector form reads

$$\vec{a} = -\nabla\Phi = \frac{\partial\Phi}{\partial R} \vec{e}_R + \frac{1}{R} \frac{\partial\Phi}{\partial\phi} \vec{e}_\phi + \frac{\partial\Phi}{\partial z} \vec{e}_z$$

Combining this with the above we see that

$$\begin{aligned} \dot{v}_R &= -\frac{\partial\Phi}{\partial R} + \frac{v_\phi^2}{R} \\ \dot{v}_\phi &= -\frac{1}{R} \frac{\partial\Phi}{\partial\phi} + \frac{v_R v_\phi}{R} \\ \dot{v}_z &= -\frac{\partial\Phi}{\partial z} \end{aligned}$$

which allows us to write the **CBE in cylindrical coordinates** as

$$\boxed{\frac{\partial f}{\partial t} + v_R \frac{\partial f}{\partial R} + \frac{v_\phi}{R} \frac{\partial f}{\partial\phi} + v_z \frac{\partial f}{\partial z} + \left[\frac{v_\phi^2}{R} - \frac{\partial\Phi}{\partial R} \right] \frac{\partial f}{\partial v_R} - \frac{1}{R} \left[v_R v_\phi + \frac{\partial\Phi}{\partial\phi} \right] \frac{\partial f}{\partial v_\phi} - \frac{\partial\Phi}{\partial z} \frac{\partial f}{\partial v_z} = 0}$$

The **Jeans equations** follow from multiplication with v_R , v_ϕ , and v_z and integrating over velocity space. Note that the **cylindrical symmetry** requires that all derivatives with respect to ϕ vanish. The remaining terms are:

$$\begin{aligned}
\int v_R \frac{\partial f}{\partial t} d^3\vec{v} &= \frac{\partial}{\partial t} \int v_R f d^3\vec{v} = \frac{\partial(\rho \langle v_R \rangle)}{\partial t} \\
\int v_R^2 \frac{\partial f}{\partial R} d^3\vec{v} &= \frac{\partial}{\partial R} \int v_R^2 f d^3\vec{v} = \frac{\partial(\rho \langle v_R^2 \rangle)}{\partial R} \\
\int v_R v_z \frac{\partial f}{\partial z} d^3\vec{v} &= \frac{\partial}{\partial z} \int v_R v_z f d^3\vec{v} = \frac{\partial(\rho \langle v_R v_z \rangle)}{\partial z} \\
\int \frac{v_R v_\phi^2}{R} \frac{\partial f}{\partial v_R} d^3\vec{v} &= \frac{1}{R} \left[\int \frac{\partial(v_R v_\phi^2 f)}{\partial v_R} d^3\vec{v} - \int \frac{\partial(v_R v_\phi^2)}{\partial v_R} f d^3\vec{v} \right] = -\rho \frac{\langle v_\phi^2 \rangle}{R} \\
\int v_R \frac{\partial \Phi}{\partial R} \frac{\partial f}{\partial v_R} d^3\vec{v} &= \frac{\partial \Phi}{\partial R} \left[\int \frac{\partial(v_R f)}{\partial v_R} d^3\vec{v} - \int \frac{\partial v_R}{\partial v_R} f d^3\vec{v} \right] = -\rho \frac{\partial \Phi}{\partial R} \\
\int \frac{v_R^2 v_\phi}{R} \frac{\partial f}{\partial v_\phi} d^3\vec{v} &= \frac{1}{R} \left[\int \frac{\partial(v_R^2 v_\phi f)}{\partial v_\phi} d^3\vec{v} - \int \frac{\partial(v_R^2 v_\phi)}{\partial v_\phi} f d^3\vec{v} \right] = -\rho \frac{\langle v_R^2 \rangle}{R} \\
\int v_R \frac{\partial \Phi}{\partial z} \frac{\partial f}{\partial v_z} d^3\vec{v} &= \frac{\partial \Phi}{\partial z} \left[\int \frac{\partial(v_R f)}{\partial v_z} d^3\vec{v} - \int \frac{\partial v_z}{\partial v_z} f d^3\vec{v} \right] = 0
\end{aligned}$$

Working out the similar terms for the other Jeans equations we finally obtain the **Jeans Equations in Cylindrical Coordinates**:

$$\boxed{
\begin{aligned}
\frac{\partial(\rho \langle v_R \rangle)}{\partial t} + \frac{\partial(\rho \langle v_R^2 \rangle)}{\partial R} + \frac{\partial(\rho \langle v_R v_z \rangle)}{\partial z} + \rho \left[\frac{\langle v_R^2 \rangle - \langle v_\phi^2 \rangle}{R} + \frac{\partial \Phi}{\partial R} \right] &= 0 \\
\frac{\partial(\rho \langle v_\phi \rangle)}{\partial t} + \frac{\partial(\rho \langle v_R v_\phi \rangle)}{\partial R} + \frac{\partial(\rho \langle v_\phi v_z \rangle)}{\partial z} + 2\rho \frac{\langle v_R v_\phi \rangle}{R} &= 0 \\
\frac{\partial(\rho \langle v_z \rangle)}{\partial t} + \frac{\partial(\rho \langle v_R v_z \rangle)}{\partial R} + \frac{\partial(\rho \langle v_z^2 \rangle)}{\partial z} + \rho \left[\frac{\langle v_R v_z \rangle}{R} + \frac{\partial \Phi}{\partial z} \right] &= 0
\end{aligned}
}$$

These are 3 equations with 9 unknowns, which can only be solved if we make additional assumptions. In particular, one often makes the following assumptions:

- 1 System is static $\Rightarrow$ the $\frac{\partial}{\partial t}$ -terms are zero and $\langle v_R \rangle = \langle v_z \rangle = 0$.
- 2 Velocity dispersion tensor is diagonal $\Rightarrow \langle v_i v_j \rangle = 0$ (if $i \neq j$).
- 3 Meridional isotropy $\Rightarrow \langle v_R^2 \rangle = \langle v_z^2 \rangle = \sigma_R^2 = \sigma_z^2 \equiv \sigma^2$.

Under these assumptions we have 3 unknowns left: $\langle v_\phi \rangle$, $\langle v_\phi^2 \rangle$, and σ^2 , and the Jeans equations reduce to

$$\boxed{\begin{aligned}\frac{\partial(\rho\sigma^2)}{\partial R} + \rho \left[\frac{\sigma^2 - \langle v_\phi^2 \rangle}{R} + \frac{\partial\Phi}{\partial R} \right] &= 0 \\ \frac{\partial(\rho\sigma^2)}{\partial z} + \rho \frac{\partial\Phi}{\partial z} &= 0\end{aligned}}$$

Since we now only have two equations left, the system is still not closed. If from the surface brightness we can estimate the mass density, $\rho(R, z)$, and hence (using the Poisson equation) the potential $\Phi(R, z)$, we can solve the second of these Jeans equations for the **meridional velocity dispersion**:

$$\sigma^2(R, z) = \frac{1}{\rho} \int_z^\infty \rho \frac{\partial\Phi}{\partial z} dz$$

and the first Jeans equation then gives the **mean square azimuthal velocity** $\langle v_\phi^2 \rangle = \langle v_\phi \rangle^2 + \sigma_\phi^2$:

$$\langle v_\phi^2 \rangle(R, z) = \sigma^2(R, z) + R \frac{\partial\Phi}{\partial R} + \frac{R}{\rho} \frac{\partial(\rho\sigma^2)}{\partial R}$$

Thus, although $\langle v_\phi^2 \rangle$ is uniquely specified by the Jeans equations, we don't know how it splits in the actual **azimuthal streaming**, $\langle v_\phi \rangle$, and the **azimuthal dispersion**, σ_ϕ^2 . Additional assumptions are needed for this.

A similar analysis, but for a spherically symmetric system, using the spherical coordinate system (r, θ, ϕ) , gives the following **Jeans equations in Spherical Symmetry**

$$\begin{aligned} \frac{\partial(\rho\langle v_r \rangle)}{\partial t} + \frac{\partial(\rho\langle v_r^2 \rangle)}{\partial r} + \frac{\rho}{r} [2\langle v_r^2 \rangle - \langle v_\theta^2 \rangle - \langle v_\phi^2 \rangle] + \rho \frac{\partial \Phi}{\partial r} &= 0 \\ \frac{\partial(\rho\langle v_\theta \rangle)}{\partial t} + \frac{\partial(\rho\langle v_r v_\theta \rangle)}{\partial r} + \frac{\rho}{r} [3\langle v_r v_\theta \rangle + (\langle v_\theta^2 \rangle - \langle v_\phi^2 \rangle) \cot \theta] &= 0 \\ \frac{\partial(\rho\langle v_\phi \rangle)}{\partial t} + \frac{\partial(\rho\langle v_r v_\phi \rangle)}{\partial r} + \frac{\rho}{r} [3\langle v_r v_\phi \rangle + 2\langle v_\theta v_\phi \rangle \cot \theta] &= 0 \end{aligned}$$

If we now make the additional assumptions that the system is **static** and that also the *kinematic* properties of the system are spherically symmetric, then there can be *no streaming motions* and all mixed second-order moments vanish. Consequently, the **velocity dispersion tensor** is diagonal with $\sigma_\theta^2 = \sigma_\phi^2$. Under these assumptions only one of the three Jeans equations remains:

$$\frac{\partial(\rho\sigma_r^2)}{\partial r} + \frac{2\rho}{r} [\sigma_r^2 - \sigma_\theta^2] + \rho \frac{\partial \Phi}{\partial r} = 0$$

Notice that this single equation still contains two unknown, $\sigma_r^2(r)$ and $\sigma_\theta^2(r)$ (if we assume that the density and potential are known), and can thus not be solved.

It is useful to define the **anisotropy parameter**

$$\beta(r) \equiv 1 - \frac{\sigma_\theta^2(r) + \sigma_\phi^2(r)}{2\sigma_r^2(r)} = 1 - \frac{\sigma_\theta^2(r)}{\sigma_r^2(r)}$$

where the second equality only holds under the assumption that the kinematics are spherically symmetric.

With β thus defined the (spherical) Jeans equation can be written as

$$\frac{1}{\rho} \frac{\partial(\rho\langle v_r^2 \rangle)}{\partial r} + 2 \frac{\beta\langle v_r^2 \rangle}{r} = - \frac{d\Phi}{dr}$$

If we now use that $d\Phi/dr = GM(r)/r$, we can write the following expression for the

enclosed (dynamical) mass:

$$M(r) = -\frac{r\langle v_r^2 \rangle}{G} \left[\frac{d \ln \rho}{d \ln r} + \frac{d \ln \langle v_r^2 \rangle}{d \ln r} + 2\beta \right]$$

Hence, if we know $\rho(r)$, $\langle v_r^2 \rangle(r)$, and $\beta(r)$, we can use the **spherical Jeans equation** to infer the mass profile $M(r)$.

Consider an external, spherical galaxy. Observationally, we can measure the projected **surface brightness** profile, $\Sigma(R)$, which is related to the 3D **luminosity density** $\nu(r) = \rho(r)/\Upsilon(r)$

$$\Sigma(R) = 2 \int_R^\infty \frac{\nu r dr}{\sqrt{r^2 - R^2}}$$

with $\Upsilon(r)$ the **mass-to-light ratio**. Similarly, the **line-of-sight velocity dispersion**, $\sigma_p^2(R)$, which can be inferred from spectroscopy, is related to both $\langle v_r^2 \rangle(r)$ and $\beta(r)$ according to (see Fig. 19)

$$\begin{aligned} \Sigma(R)\sigma_p^2(R) &= 2 \int_R^\infty \langle (v_r \cos \alpha - v_\theta \sin \alpha)^2 \rangle \frac{\nu r dr}{\sqrt{r^2 - R^2}} \\ &= 2 \int_R^\infty (\langle v_r^2 \rangle \cos^2 \alpha + \langle v_\theta^2 \rangle \sin^2 \alpha) \frac{\nu r dr}{\sqrt{r^2 - R^2}} \\ &= 2 \int_R^\infty \left(1 - \beta \frac{R^2}{r^2} \right) \frac{\nu \langle v_r^2 \rangle r dr}{\sqrt{r^2 - R^2}} \end{aligned}$$

The 3D luminosity density is trivially obtained from the observed $\Sigma(R)$ using the **Abel transform**

$$\nu(r) = -\frac{1}{\pi} \int_r^\infty \frac{d\Sigma}{dR} \frac{dR}{\sqrt{R^2 - r^2}}$$

In general, we have three unknowns: $M(r)$ [or equivalently $\rho(r)$ or $\Upsilon(r)$], $\langle v_r^2 \rangle(r)$ and $\beta(r)$. With our two observables $\Sigma(R)$ and $\sigma_p^2(R)$, these can only be determined if we make additional assumptions.

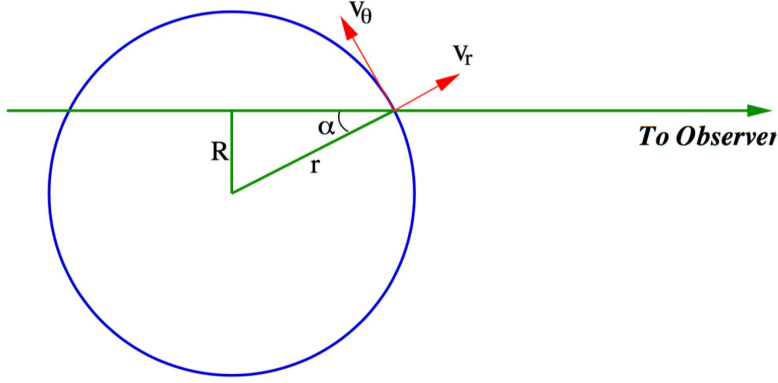


Figure 19: *Geometry related to projection*

EXAMPLE 1: Assume isotropy: $\beta(r) = 0$. In this case we can use the **Abel transform** to obtain

$$\nu(r) \langle v_r^2 \rangle(r) = -\frac{1}{\pi} \int_r^\infty \frac{d(\Sigma \sigma_p^2)}{dR} \frac{dR}{\sqrt{R^2 - r^2}}$$

and the enclosed mass follows from the Jeans equation

$$M(r) = -\frac{r \langle v_r^2 \rangle}{G} \left[\frac{d \ln \nu}{d \ln r} + \frac{d \ln \langle v_r^2 \rangle}{d \ln r} \right]$$

Note that the first term uses the *luminosity* density $\nu(r)$ rather than the *mass* density $\rho(r)$. This is because σ_p^2 is weighted by light rather than mass.

The mass-to-light ratio now follows from

$$\Upsilon(r) = \frac{M(r)}{4\pi \int_0^r \nu(r) r^2 dr}$$

which can be used to constrain the mass of a potential dark matter halo or central supermassive black hole (but always under assumption that the system is isotropic).

EXAMPLE 2: Assume a constant mass-to-light ratio: $\Upsilon(r) = \Upsilon_0$. In this case the luminosity density $\nu(r)$ immediately yields the enclosed mass:

$$M(r) = 4\pi \Upsilon_0 \int_0^r \nu(r) r^2 dr$$

We can now use the **spherical Jeans Equation** to write $\beta(r)$ in terms of $M(r)$, $\nu(r)$ and $\langle v_r^2 \rangle(r)$. Substituting this in the equation for $\Sigma(R)\sigma_p^2(R)$ yields a solution for $\langle v_r^2 \rangle(r)$, and thus for $\beta(r)$. As long as $\beta(r) \leq 1$ the model is said to be **self-consistent** within the context of the Jeans equations.

Almost always, radically different models (based on radically different assumptions) can be constructed, that are all consistent with the data and the Jeans equations. This is often referred to as the **mass-anisotropy degeneracy**. Note, however, that none of these models need to be physical: they can still have $f < 0$.

CHAPTER 16

Orbit Theory

The ‘building-blocks’ of collisionless systems, such as galaxies and dark matter halos, are orbits. In this Chapter, we very briefly highlight a few aspects of orbit theory. A more detailed account of orbit theory in the context of astrophysical systems can be found in the excellent textbook ”*Galactic Dynamics*” by Binney & Tremaine.

Integrals of Motion: An integral of motion is a function $I(\vec{x}, \vec{v})$ of the phase-space coordinates that is constant along all orbits, i.e.,

$$I[\vec{x}(t_1), \vec{v}(t_1)] = I[\vec{x}(t_2), \vec{v}(t_2)]$$

for any t_1 and t_2 . The value of the integral of motion can be the same for different orbits. Note that an integral of motion can not depend on time. Orbits can have from zero to five integrals of motion. If the Hamiltonian does not depend on time, then energy is always an integral of motion.

Integrals of motion come in two kinds:

- **Isolating Integrals of Motion:** these reduce the dimensionality of the particle’s trajectory in 6-dimensional phase-space by one. Therefore, an orbit with n isolating integrals of motion is restricted to a $6 - n$ dimensional manifold in 6-dimensional phase-space. Energy is always an isolating integral of motion.
- **Non-Isolating Integrals of Motion:** these are integrals of motion that do *not* reduce the dimensionality of the particle’s trajectory in phase-space. They are of essentially no practical value for the dynamics of the system.

Orbits: If in a system with n degrees of freedom a particular orbit admits n independent isolating integrals of motion, the orbit is said to be **regular**, and its corresponding trajectory $\Gamma(t)$ is confined to a $2n - n = n$ dimensional manifold in phase-space. Topologically this manifold is called an **invariant torus** (or torus for short), and is uniquely specified by the n isolating integrals. A regular orbit has n fundamental frequencies, ω_i , with which it circulates or librates in its n -dimensional manifold. If two or more of these frequencies are commensurable (i.e., $l\omega_i + m\omega_j = 0$ with l and m integers), then the orbit is a **resonant orbit**, and has a dimensionality

that is one lower than that of the non-resonant regular orbits (i.e., $l\omega_i + m\omega_j$ is an extra isolating integral of motion). Orbits with fewer than n isolating integrals of motion are called **irregular** or **stochastic**.

Every spherical potential admits at least four isolating integrals of motion, namely energy, E , and the three components of the angular momentum vector $\vec{L}$. Orbits in a flattened, axisymmetric potential frequently (but not always) admit three isolating integrals of motion: E , L_z (where the z -axis is the system's symmetry axis), and a **non-classical** third integral I_3 (the integral is called non-classical since there is no analytical expression of I_3 as function of the phase-space variables).

Since an integral of motion, $I(\vec{x}, \vec{v})$ is constant along an orbit, we have that

$$\frac{dI}{dt} = \frac{\partial I}{\partial x_i} \frac{dx_i}{dt} + \frac{\partial I}{\partial v_i} \frac{dv_i}{dt} = \vec{v} \cdot \nabla I - \nabla \Phi \cdot \frac{\partial I}{\partial \vec{v}} = 0$$

Compare this to the **CBE** for a steady-state (static) system:

$$\vec{v} \cdot \nabla f - \nabla \Phi \cdot \frac{\partial f}{\partial \vec{v}} = 0$$

Thus the condition for I to be an **integral of motion** is identical with the condition for I to be a **steady-state solution of the CBE**. This implies the following:

Jeans Theorem: *Any steady-state solution of the CBE depends on the phase-space coordinates only through integrals of motion. Any function of these integrals is a steady-state solution of the CBE.*

Strong Jeans Theorem: *The DF of a steady-state system in which almost all orbits are regular can be written as a function of the independent isolating integrals of motion.*

Hence, the DF of any steady-state spherical system can be expressed as $f = f(E, \vec{L})$. If the system is spherically symmetric in *all* its properties, then $f = f(E, L^2)$, i.e., the DF can only depend on the magnitude of the angular momentum vector, not on its direction.

An even simpler case to consider is the one in which $f = f(E)$: Since $E = \Phi(\vec{r}) + \frac{1}{2}[v_r^2 + v_\theta^2 + v_\phi^2]$ we have that

$$\begin{aligned}\langle v_r^2 \rangle &= \frac{1}{\rho} \int dv_r dv_\theta dv_\phi v_r^2 f \left(\Phi + \frac{1}{2}[v_r^2 + v_\theta^2 + v_\phi^2] \right) \\ \langle v_\theta^2 \rangle &= \frac{1}{\rho} \int dv_r dv_\theta dv_\phi v_\theta^2 f \left(\Phi + \frac{1}{2}[v_r^2 + v_\theta^2 + v_\phi^2] \right) \\ \langle v_\phi^2 \rangle &= \frac{1}{\rho} \int dv_r dv_\theta dv_\phi v_\phi^2 f \left(\Phi + \frac{1}{2}[v_r^2 + v_\theta^2 + v_\phi^2] \right)\end{aligned}$$

Since these equations differ only in the labelling of one of the variables of integration, it is immediately evident that $\langle v_r^2 \rangle = \langle v_\theta^2 \rangle = \langle v_\phi^2 \rangle$. Hence, assuming that $f = f(E)$ is identical to assuming that the system is **isotropic** (and thus $\beta(r) = 0$). And since

$$\langle v_i \rangle = \frac{1}{\rho} \int dv_r dv_\theta dv_\phi v_i f \left(\Phi + \frac{1}{2}[v_r^2 + v_\theta^2 + v_\phi^2] \right)$$

it is also immediately evident that $\langle v_r \rangle = \langle v_\theta \rangle = \langle v_\phi \rangle = 0$. Thus, a system with $f = f(E)$ has no net sense of rotation.

The more general $f(E, L^2)$ models typically are **anisotropic**. Models with $0 < \beta \leq 1$ are radially anisotropic. In the extreme case of $\beta = 1$ all orbits are purely radial and $f = g(E) \delta(L)$, with $g(E)$ some function of energy. Tangentially anisotropic models have $\beta < 0$, with $\beta = -\infty$ corresponding to a model in which all orbits are circular. In that case $f = g(E) \delta[L - L_{\max}(E)]$, where $L_{\max}(E)$ is the maximum angular momentum for energy E . Another special case is the one in which $\beta(r) = \beta$ is constant; such models have $f = g(E) L^{-2\beta}$.

Next we consider **axisymmetric** systems. If we only consider systems for which most orbits are regular, then the **strong Jeans Theorem** states that, in the most general case, $f = f(E, L_z, I_3)$. For a static, axisymmetric system

$$\langle v_R \rangle = \langle v_z \rangle = 0 \quad \langle v_R v_\phi \rangle = \langle v_z v_\phi \rangle = 0$$

but note that, in this general case, $\langle v_R v_z \rangle \neq 0$; Hence, in general, in a three-integral model with $f = f(E, L_z, I_3)$ the **velocity ellipsoid** is not aligned with (R, ϕ, z) , and the velocity dispersion tensor contains four unknowns: $\langle v_R^2 \rangle$, $\langle v_\phi^2 \rangle$, $\langle v_z^2 \rangle$, and $\langle v_R v_z \rangle$. In this case there are two non-trivial Jeans Equations:

$$\boxed{\boxed{\begin{aligned}\frac{\partial(\rho\langle v_R^2 \rangle)}{\partial R} + \frac{\partial(\rho\langle v_R v_z \rangle)}{\partial z} + \rho \left[\frac{\langle v_R^2 \rangle - \langle v_\phi^2 \rangle}{R} + \frac{\partial\Phi}{\partial R} \right] &= 0 \\ \frac{\partial(\rho\langle v_R v_z \rangle)}{\partial R} + \frac{\partial(\rho\langle v_z^2 \rangle)}{\partial z} + \rho \left[\frac{\langle v_R v_z \rangle}{R} + \frac{\partial\Phi}{\partial z} \right] &= 0\end{aligned}}}$$

which clearly doesn't suffice to solve for the four unknowns (modelling three-integral axisymmetric systems is best done using the Schwarzschild orbit superposition technique). To make progress with Jeans modeling, one has to make additional assumptions. A typical assumption is that the DF has the **two-integral form** $f = f(E, L_z)$.

In that case, $\langle v_R v_z \rangle = 0$ [velocity ellipsoid now *is* aligned with (R, ϕ, z)] and $\langle v_R^2 \rangle = \langle v_z^2 \rangle$ (see Binney & Tremaine 2008), so that the **Jeans equations** reduce to

$$\boxed{\boxed{\begin{aligned}\frac{\partial(\rho\langle v_R^2 \rangle)}{\partial R} + \rho \left[\frac{\langle v_R^2 \rangle - \langle v_\phi^2 \rangle}{R} + \frac{\partial\Phi}{\partial R} \right] &= 0 \\ \frac{\partial(\rho\langle v_z^2 \rangle)}{\partial z} + \rho \frac{\partial\Phi}{\partial z} &= 0\end{aligned}}}$$

which is a closed set for the two unknowns $\langle v_R^2 \rangle$ ($= \langle v_z^2 \rangle$) and $\langle v_\phi^2 \rangle$. Note, however, that the Jeans equations provide no information about how $\langle v_\phi^2 \rangle$ splits in streaming and random motions. In practice one often writes that

$$\langle v_\phi \rangle = k [\langle v_\phi^2 \rangle - \langle v_R^2 \rangle]$$

with k a free parameter. When $k = 1$ the azimuthal dispersion is $\sigma_\phi^2 \equiv \langle v_\phi^2 \rangle - \langle v_\phi \rangle^2 = \sigma_R^2 = \sigma_z^2$ everywhere. Such models are called **oblate isotropic rotators**.

CHAPTER 17

Collisions & Encounters of Collisionless Systems

Consider an encounter between two collisionless N-body systems (i.e., dark matter halos or galaxies): a perturber P and a system S . Let q denote a particle of S and let b be the impact parameter, v_∞ the initial speed of the encounter, and R_0 the distance of closest approach (see Fig. 20).

Typically what happens in an encounter is that **orbital energy** (of P wrt S) is **converted into random motion energy** of the constituent particles of P and S (i.e., q gains kinetic energy wrt S).

The **velocity impulse** $\Delta\vec{v}_q = \int \vec{g}(t) dt$ of q due to the gravitational field $\vec{g}(t)$ from P decreases with increasing v_∞ (simply because Δt will be shorter). Consequently, when v_∞ increases, less and less orbital energy is transferred to random motion, and there is a critical velocity, v_{crit} , such that

$$\begin{aligned} v_\infty > v_{\text{crit}} &\Rightarrow S \text{ and } P \text{ escape from each other} \\ v_\infty < v_{\text{crit}} &\Rightarrow S \text{ and } P \text{ merge together} \end{aligned}$$

There are only two cases in which we can calculate the outcome of the encounter analytically:

- **high speed encounter** ($v_\infty \gg v_{\text{crit}}$). In this case the encounter is said to be **impulsive** and one can use the impulsive approximation to compute its outcome.
- **large mass ratio** ($M_P \ll M_S$). In this case one can use the treatment of **dynamical friction** to describe how P loses orbital energy and angular momentum to S .

In all other cases, one basically has to resort to numerical simulations to study the outcome of the encounter. In what follows we present treatments of first the impulse approximation and then dynamical friction.

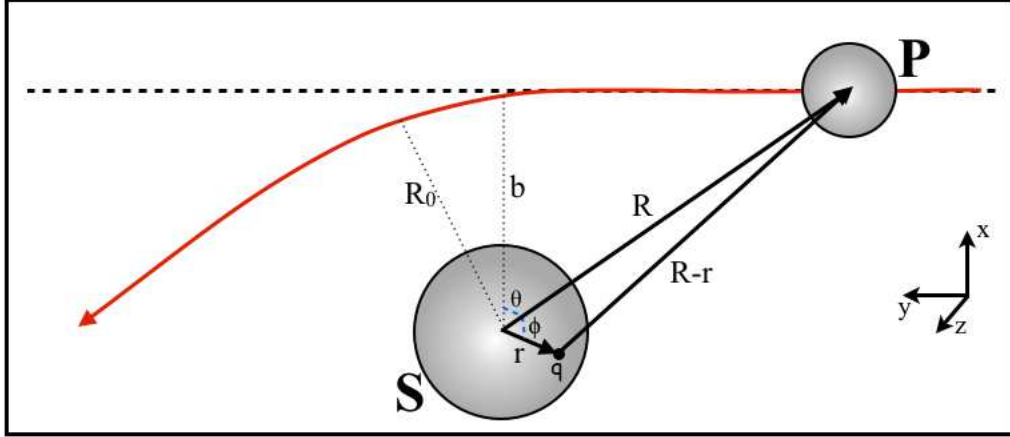


Figure 20: *Schematic illustration of an encounter with impact parameter b between a perturber P and a subject S .*

Impulse Approximation: In the limit where the encounter velocity v_∞ is much larger than the internal velocity dispersion of S , the change in the internal energy of S can be approximated analytically. The reason is that, in a high-speed encounter, the time-scale over which the tidal forces from P act on q is much shorter than the dynamical time of S (or q). Hence, we may consider q to be stationary (fixed wrt S) during the encounter. Consequently, q only experiences a change in its **kinetic** energy, while its potential energy remains unchanged:

$$\Delta E_q = \frac{1}{2}(\vec{v} + \Delta\vec{v})^2 - \frac{1}{2}\vec{v}^2 = \vec{v} \cdot \Delta\vec{v} + \frac{1}{2}|\Delta\vec{v}|^2$$

We are interested in $\Delta E_S = \sum_q \Delta E_q$, where the summation is over all its constituent particles:

$$\Delta E_S = \int \Delta E(\vec{r}) \rho(r) d^3\vec{r} \simeq \frac{1}{2} \int |\Delta\vec{v}|^2 \rho(r) d^3\vec{r}$$

where we have used that, because of symmetry, the integral

$$\int \vec{v} \cdot \Delta\vec{v} \rho(r) d^3\vec{r} \simeq 0$$

In the large v_∞ limit, we have that the distance of closest approach $R_0 \rightarrow b$, and the velocity of P wrt S is $v_P(t) \simeq v_\infty \vec{e}_y \equiv v_P \vec{e}_y$. Consequently, we have that

$$\vec{R}(t) = (b, v_P t, 0)$$

Let $\vec{r}$ be the position vector of q wrt S and adopt the **distant encounter approximation**, which means that $b \gg \max[R_S, R_P]$, where R_S and R_P are the sizes of S and P , respectively. This means that we may treat P as a point mass M_P , so that

$$\Phi_P(\vec{r}) = -\frac{GM_P}{|\vec{r} - \vec{R}|}$$

Using geometry, and defining ϕ as the angle between $\vec{r}$ and $\vec{R}$, we have that

$$|\vec{r} - \vec{R}|^2 = (R - r \cos \phi)^2 + (r \sin \phi)^2$$

so that

$$|\vec{r} - \vec{R}| = \sqrt{R^2 - 2rR \cos \phi + r^2}$$

Next we use the series expansion

$$\frac{1}{\sqrt{1+x}} = 1 - \frac{1}{2}x + \frac{1}{2} \frac{3}{4}x^2 - \frac{1}{2} \frac{3}{4} \frac{5}{6}x^3 + \dots$$

to write

$$\frac{1}{|\vec{r} - \vec{R}|} = \frac{1}{R} \left[1 - \frac{1}{2} \left(-2 \frac{r}{R} \cos \phi + \frac{r^2}{R^2} \right) + \frac{3}{8} \left(-2 \frac{r}{R} \cos \phi + \frac{r^2}{R^2} \right)^2 + \dots \right]$$

Substitution in the expression for the potential of P yields

$$\Phi_P(\vec{r}) = -\frac{GM_P}{R} - \frac{GM_P r}{R^2} \cos \phi - \frac{GM_P r^2}{R^3} \left(\frac{3}{2} \cos^2 \phi - \frac{1}{2} \right) + \mathcal{O}[(r/R)^3]$$

- The first term on rhs is a constant, not yielding any force (i.e., $\nabla_r \Phi_P = 0$).
- The second term on the rhs describes how the center of mass of S changes its velocity due to the encounter with P .
- The third term on the rhs corresponds to the **tidal force** per unit mass and is the term of interest to us.

It is useful to work in a rotating coordinate frame (x', y', z') centered on S and with the x' -axis pointing towards the instantaneous location of P , i.e., x' points along $\vec{R}(t)$. Hence, we have that $x' = r' \cos \phi$, where $r'^2 = x'^2 + y'^2 + z'^2$. In this new coordinate frame, we can express the third term of $\Phi_P(\vec{r})$ as

$$\begin{aligned}
\Phi_3(\vec{r}) &= -\frac{GM_P}{R^3} \left(\frac{3}{2} r'^2 \cos^2 \phi - \frac{1}{2} r'^2 \right) \\
&= -\frac{GM_P}{R^3} \left(x'^2 - \frac{1}{2} y'^2 - \frac{1}{2} z'^2 \right)
\end{aligned}$$

Hence, the tidal force is given by

$$\vec{F}'_{\text{tid}}(\vec{r}) \equiv -\nabla \Phi_P = \frac{GM_P}{R^3} (2x', -y', -z')$$

We can relate the components of $\vec{F}'_{\text{tid}}$ to those of the corresponding tidal force, $\vec{F}_{\text{tid}}$ in the (x, y, z) -coordinate system using

$$\begin{aligned}
x' &= x \cos \theta - y \sin \theta & F_x &= F_{x'} \cos \theta + F_{y'} \sin \theta \\
y' &= x \sin \theta + y \cos \theta & F_y &= -F_{x'} \sin \theta + F_{y'} \cos \theta \\
z' &= z & F_z &= F_{z'}
\end{aligned}$$

where θ is the angle between the x and x' axes, with $\cos \theta = b/R$ and $\sin \theta = v_P t/R$. After some algebra one finds that

$$\begin{aligned}
F_x &= \frac{GM_P}{R^3} [x(2 - 3 \sin^2 \theta) - 3y \sin \theta \cos \theta] \\
F_y &= \frac{GM_P}{R^3} [y(2 - 3 \cos^2 \theta) - 3x \sin \theta \cos \theta] \\
F_z &= -\frac{GM_P}{R^3} z
\end{aligned}$$

Using these, we have that

$$\Delta v_x = \int \frac{dv_x}{dt} dt = \int F_x dt = \int_{-\pi/2}^{\pi/2} F_x \frac{dt}{d\theta} d\theta$$

with similar expressions for Δv_y and Δv_z . Using that $\theta = \tan^{-1}(v_P t/b)$ one has that $dt/d\theta = b/(v_P \cos^2 \theta)$. Substituting the above expressions for the tidal force, and using that $R = b/\cos \theta$, one finds, after some algebra, that

$$\Delta \vec{v} = (\Delta v_x, \Delta v_y, \Delta v_z) = \frac{2GM_P}{v_P b^2} (x, 0, -z)$$

Substitution in the expression for ΔE_S yields

$$\Delta E_S = \frac{1}{2} \int |\Delta \vec{v}|^2 \rho(r) d^3 \vec{r} = \frac{2 G^2 M_P^2}{v_P^2 b^4} M_S \langle x^2 + z^2 \rangle$$

Under the assumption that S is spherically symmetric we have that $\langle x^2 + z^2 \rangle = \frac{2}{3} \langle x^2 + y^2 + z^2 \rangle = \frac{2}{3} \langle r^2 \rangle$ and we obtain the final expression for the energy increase of S as a consequence of the impulsive encounter with P :

$$\boxed{\Delta E_S = \frac{4}{3} G^2 M_S \left(\frac{M_P}{v_P} \right)^2 \frac{\langle r^2 \rangle}{b^4}}$$

This derivation, which is originally due to Spitzer (1958), is surprisingly accurate for encounters with $b > 5 \max[R_P, R_S]$, even for relatively slow encounters with $v_\infty \sim \sigma_S$. For smaller impact parameters one has to make a correction (see **Galaxy Formation and Evolution** by Mo, van den Bosch & White 2010 for details).

The **impulse approximation** shows that high-speed encounters can pump energy into the systems involved. This energy is tapped from the orbital energy of the two systems wrt each other. Note that $\Delta E_S \propto b^{-4}$, so that close encounters are far more important than distant encounters.

Let $E_b \propto GM_S/R_S$ be the **binding energy** of S . Then, it is tempting to postulate that if $\Delta E_S > E_b$ the impulsive encounter will lead to the **tidal disruption** of S . However, this is not at all guaranteed. What is important for disruption is how that energy ΔE_S is distributed over the constituent particles of S . Since $\Delta E \propto r^2$, particles in the outskirts of S typically gain much more energy than those in the center. However, particles in the outskirts are least bound, and thus require the least amount of energy to become unbound. Particles that are strongly bound (with small r) typically gain very little energy. As a consequence, a significant fraction of the mass can remain bound even if $\Delta E_S \gg E_b$ (see van den Bosch et al., 2019, MNRAS, **474**, 3043 for details).

After the encounter, S has gained **kinetic energy** (in the amount of ΔE_S), but its **potential energy** has remained unchanged (recall, this is the assumption that underlies the impulse approximation). As a consequence, after the encounter S will no longer be in **virial equilibrium**; S will have to readjust itself to re-establish virial equilibrium.

Let K_0 and E_0 be the initial (pre-encounter) kinetic and total energy of S . The **virial theorem** ensures that $E_0 = -K_0$. The encounter causes an increase of (kinetic) energy, so that $K_0 \rightarrow K_0 + \Delta E_S$ and $E_0 \rightarrow E_0 + \Delta E_S$. After S has re-established virial equilibrium, we have that $K_1 = -E_1 = -(E_0 + \Delta E_S) = K_0 - \Delta E_S$. Thus, we see that virialization after the encounter changes the kinetic energy of S from $K_0 + \Delta E_S$ to $K_0 - \Delta E_S$! The gravitational energy after the encounter is $W_1 = 2E_1 = 2E_0 + 2\Delta E_S = W_0 + 2\Delta E_S$, which is less negative than before the encounter. Using the definition of the **gravitational radius** (see Chapter 18), $r_g = GM_S^2/|W|$, from which it is clear that the (gravitational) radius of S increases due to the impulsive encounter. Note that here we have ignored the complication coming from the fact that the injection of energy ΔE_S may result in unbinding some of the mass of S .

Dynamical Friction: Consider the motion of a subject mass M_S through a medium of individual particles of mass $m \ll M_S$. The subject mass M_S experiences a "drag force", called dynamical friction, which transfers orbital energy and angular momentum from M_S to the sea of particles of mass m .

There are three different "views" of dynamical friction:

1. Dynamical friction arises from **two-body encounters** between the subject mass and the particles of mass m , which drives the system towards **equipartition**. i.e., towards $\frac{1}{2}M_S v_S^2 = \frac{1}{2}m \langle v_m^2 \rangle$. Since $M_S \gg m$, the system thus evolves towards $v_S \ll v_m$ (i.e., M_S slows down).
2. Due to **gravitational focussing** the subject mass M_S creates an overdensity of particles behind its path (the "wake"). The gravitational back-reaction of this wake on M_S is what gives rise to dynamical friction and causes the subject mass to slow down.
3. The subject mass M_S causes a perturbation $\delta\Phi$ in the potential of the collection of particles of mass m . The gravitational interaction between the **response density** (the density distribution that corresponds to $\delta\Phi$ according to the Poisson equation) and the subject mass is what gives rise to dynamical friction (see Fig. 21).

Although these views are similar, there are some subtle differences. For example, according to the first two descriptions dynamical friction is a **local** effect. The

third description, on the other hand, treats dynamical friction more as a **global** effect. As we will see, there are circumstances under which these views make different predictions, and if that is the case, the third and latter view presents itself as the better one.

Chandrasekhar derived an expression for the dynamical friction force which, although it is based on a number of questionable assumptions, yields results in reasonable agreement with simulations. This so-called **Chandrasekhar dynamical friction** force is given by

$$\vec{F}_{\text{df}} = M_{\text{S}} \frac{d\vec{v}_{\text{S}}}{dt} = -\frac{4\pi G^2 M_{\text{S}}^2}{v_{\text{S}}^2} \ln \Lambda \rho(< v_{\text{S}}) \frac{\vec{v}_{\text{S}}}{v_{\text{S}}}$$

Here $\rho(< v_{\text{S}})$ is the density of particles of mass m that have a speed $v_m < v_{\text{S}}$, and $\ln \Lambda$ is called the **Coulomb logarithm**. It's value is uncertain (typically $3 \lesssim \ln \Lambda \lesssim 30$). One often approximates it as $\ln \Lambda \sim \ln(M_{\text{h}}/M_{\text{S}})$, where M_{h} is the total mass of the system of particles of mass m , but this should only be considered a very rough estimate at best. The uncertainties for the Coulomb logarithm derive from the oversimplified assumptions made by Chandrasekhar, which include that the medium through which the subject mass is moving is infinite, uniform and with an isotropic velocity distribution $f(v_m)$ for the sea of particles.

Similar to **frictional drag** in fluid mechanics, $\vec{F}_{\text{df}}$ is always pointing in the direction **opposite** of v_{S} .

Contrary to **frictional drag** in fluid mechanics, which always increases in strength when v_{S} increases, dynamical friction has a more complicated behavior: In the low- v_{S} limit, $F_{\text{df}} \propto v_{\text{S}}$ (similar to hydrodynamical drag). However, in the high- v_{S} limit one has that $F_{\text{df}} \propto v_{\text{S}}^{-2}$ (which arises from the fact that the factor $\rho(< v_{\text{S}})$ saturates).

Note that $\vec{F}_{\text{df}}$ is **independent** of the mass m of the constituent particles, and proportional to M_{S}^2 . The latter arises, within the second or third view depicted above, from the fact that the wake or response density has a mass that is proportional to M_{S} , and the gravitational force between the subject mass and the wake/response density therefore scales as M_{S}^2 .

One shortcoming of Chandrasekhar's dynamical friction description is that it treats

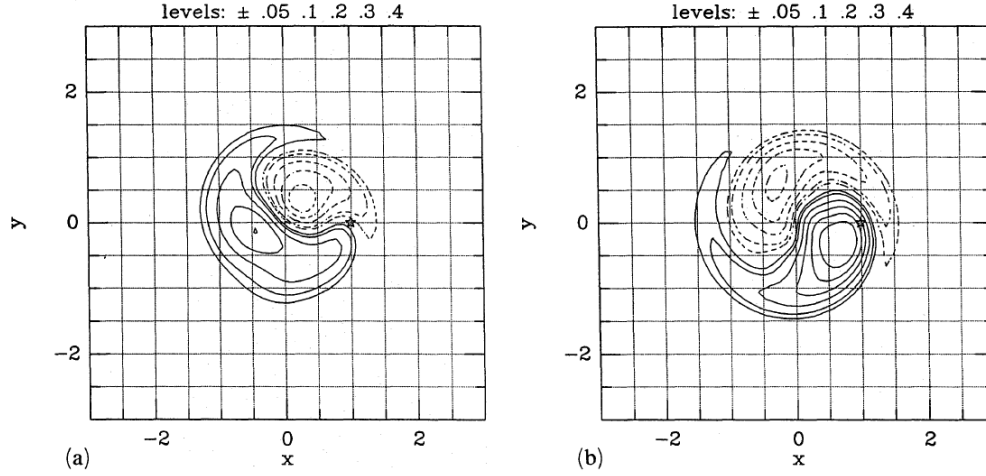


Figure 21: *Examples of the response density in a host system due to a perturber orbiting inside it. The back-reaction of this response density on the perturber causes the latter to experience dynamical friction. The position of the perturber is indicated by an asterisk. [Source: Weinberg, 1989, MNRAS, 239, 549]*

dynamical friction as a purely **local phenomenon**; it is treated as the cumulative effect of many **uncorrelated** two-body encounters between the subject mass and the individual field particles. That this local treatment is incomplete is evident from the fact that an object *A* orbiting **outside** of an *N*-body system *B* still experiences dynamical friction. This can be understood with the picture sketched under view 3 above, but not in a view that treats dynamical friction as a local phenomenon.

Orbital decay: As a consequence of dynamical friction, a subject mass M_s orbiting inside (or just outside) of a larger *N*-body system of mass $M_h > M_s$, will transfer its orbital energy and angular momentum to the constituent particles of the ‘host’ mass. As a consequence it experiences orbital decay.

Let us assume that the host mass is a **singular isothermal sphere** with density and potential given by

$$\rho(r) = \frac{V_c^2}{4\pi G r^2} \quad \Phi(r) = V_c^2 \ln r$$

where $V_c^2 = GM_h/r_h$ with r_h the radius of the host mass. If we further assume that this host mass has, at each point, an **isotropic** and **Maxwellian** velocity distribution, then

$$f(v_m) = \frac{\rho(r)}{(2\pi\sigma^2)^{3/2}} \exp\left[-\frac{v_m^2}{2\sigma^2}\right]$$

with $\sigma = V_c/\sqrt{2}$.

NOTE: the assumption of a singular isothermal sphere with an isotropic, Maxwellian velocity distribution is unrealistic, but it serves the purpose of the order-of-magnitude estimate for the orbital decay rate presented below.

Now consider a subject of mass M_S moving on a **circular orbit** ($v_S = V_c$) through this host system of mass M_h . The **Chandrasekhar dynamical friction** that this subject mass experiences is

$$F_{\text{df}} = -\frac{4\pi \ln \Lambda G^2 M_S^2 \rho(r)}{V_c^2} \left[\text{erf}(1) - \frac{2}{\sqrt{\pi}} e^{-1} \right] \simeq -0.428 \ln \Lambda \frac{GM_S^2}{r^2}$$

The subject mass has **specific angular momentum** $L = rv_S$, which it loses due to dynamical friction at a rate

$$\frac{dL}{dt} = r \frac{dv_S}{dt} = r \frac{F_{\text{df}}}{M_S} \simeq -0.428 \ln \Lambda \frac{GM_S}{r}$$

Due to this angular momentum loss, the subject mass moves to a smaller radius, while it continues to move on a circular orbit with $v_S = V_c$. Hence, the rate at which the orbital radius changes obeys

$$V_c \frac{dr}{dt} = \frac{dL}{dt} = -0.428 \ln \Lambda \frac{GM_S}{r}$$

Solving this differential equation subject to the initial condition that $r(0) = r_i$, one finds that the subject mass M_S reaches the center of the host after a time

$$t_{\text{df}} = \frac{1.17}{\ln \Lambda} \frac{r_i^2 V_c}{G M_S} = \frac{1.17}{\ln \Lambda} \left(\frac{r_i}{r_h} \right)^2 \frac{M_h}{M_S} \frac{r_h}{V_c}$$

In the case where the host system is a **virialized dark matter halo** we have that

$$\frac{r_h}{V_c} \simeq \frac{1}{10H(z)} = 0.1t_H$$

where t_H is called the **Hubble time**, and is approximately equal to the age of the Universe corresponding to redshift z (the above relation derives from the fact

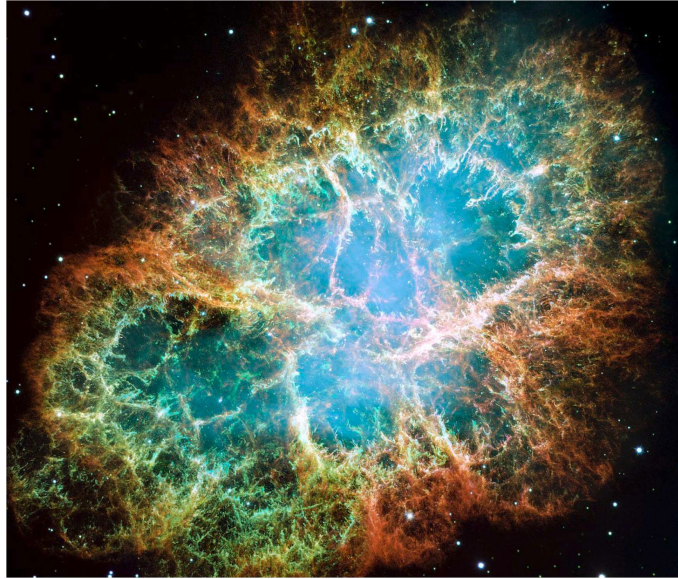
that virialized dark matter halos all have the same average density). Using that $\ln \Lambda \sim \ln(M_h/M_S)$ and assuming that the subject mass starts out from an initial radius $r_i = r_h$, we obtain a dynamical friction time

$$t_{\text{df}} = 0.12 \frac{M_h/M_S}{\ln(M_h/M_S)} t_H$$

Hence, the time t_{df} on which dynamical friction brings an object of mass M_S moving in a host of mass M_h from an initial radius of $r_i = r_h$ to $r = 0$ is shorter than the Hubble time as long as $M_S \gtrsim M_h/30$. Hence, dynamical friction is only effective for fairly massive objects, relative to the mass of the host. In fact, if you take into account that the subject mass experiences mass stripping as well (due to the tidal interactions with the host), the dynamical friction time increases by a factor 2 to 3, and $t_{\text{df}} < t_H$ actually requires that $M_S \gtrsim M_h/10$.

For a more detailed treatment of collisions and encounters of collisionless systems, see Chapter 12 of "Galaxy Formation and Evolution" by Mo, van den Bosch & White.

Part III: Radiative Processes



With the exception of meteorites, neutrinos, gravitational waves, and cosmic rays, all information about the Universe reaches us in the form of radiation. Understanding how radiation is produced, and how it interacts with matter on its way from the source to our telescopes is therefore of crucial importance for astrophysics. The following chapters give an elementary introduction into these topics.

Radiative processes is a rich topic, and one could easily devote an entire course to it. The following chapters therefore only scratch the surface of this rich topic. Readers who want to get more indepth information are referred to the following excellent textbooks

- *Radiative Processes in Astrophysics* by G. Rybicki & A. Lightman
- *Astrophysics: Decoding the Cosmos* by J. Irwin
- *The Physics of Astrophysics I. Radiation* by F. Shu
- *Theoretical Astrophysics* by M. Bartelmann

CHAPTER 18

Radiation Essentials

Spectral Energy Distribution: the radiation from a source may be characterized by its spectral energy distribution (SED), $L_\nu d\nu$, or, equivalently, $L_\lambda d\lambda$. Some texts refer to the SEDs as the *spectral luminosity* or the *spectral power*. The SED is the total energy emitted by photons in the frequency interval $[\nu, \nu + d\nu]$, and is related to the total **luminosity**, $L \equiv dE/dt$, according to

$$L = \int L_\nu d\nu = \int L_\lambda d\lambda$$

Note that $[L_\nu] = \text{erg s}^{-1} \text{ Hz}^{-1}$, while $[L] = \text{erg s}^{-1}$.

Flux: The flux, f , of a source is the radiation energy per unit time passing through a unit area

$$dL = f dA \quad [f] = \text{erg s}^{-1} \text{ cm}^{-2}$$

where A is the area. Similarly, we can also define the **spectral flux density** (or simply ‘flux density’), as the flux per unit spectral bandwidth:

$$dL_\nu = f_\nu dA \quad [f_\nu] = \text{erg s}^{-1} \text{ cm}^{-2} \text{ Hz}^{-1}$$

In radio astronomy, one typically expresses f_ν in **Jansky**, where $1\text{Jy} = 10^{-23} \text{ erg s}^{-1} \text{ cm}^{-2} \text{ Hz}^{-1}$. As with the SEDs, one may also express spectral flux densities as f_λ . Using that $\lambda = c/\nu$, and using that $f_\nu d\nu = f_\lambda d\lambda$ one has that

$$f_\nu = \frac{\lambda^2}{c} f_\lambda, \quad f_\lambda = \frac{\nu^2}{c} f_\nu$$

Luminosity and flux are related according to

$$L = 4\pi r^2 f$$

where r is the distance from the source.

Intensity: The intensity, I , also called **surface brightness** is the flux emitted in, or observed from, a solid angle $d\Omega$. The intensity is related to the flux via

$$df = I \cos \theta d\Omega$$

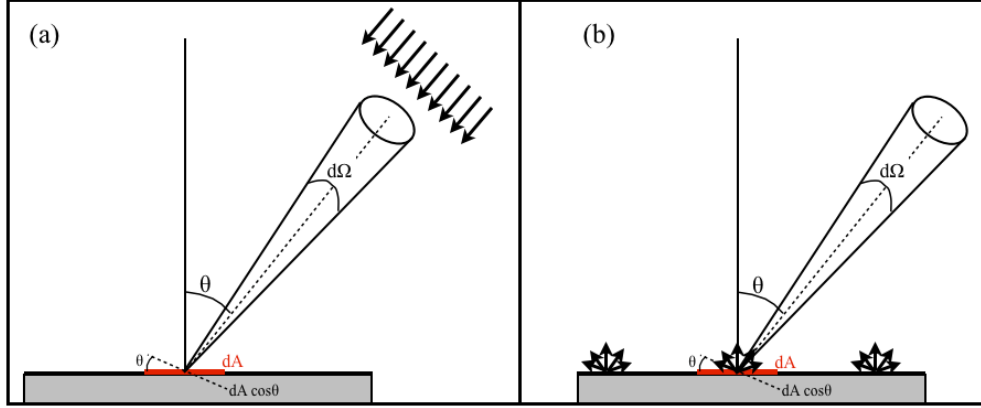


Figure 22: *Diagrams showing intensity and its dependence on direction and solid angle. Fig. (a) depicts the ‘observational view’, where dA represents an element of a detector. The arrows show incoming rays from the **center** of the source. Fig. (b) depicts the ‘emission view’, where dA represents the surface of a star. At each point on the surface, photons leave in all directions away from the surface.*

where θ is the angle between the normal of the surface area through which the flux is measured and the direction of the solid angle. The unit of intensity is $[I] = \text{erg s}^{-1} \text{cm}^{-2} \text{sr}^{-1}$. Here ‘sr’ is a **steradian**, which is the unit of solid angle measure (there are 4π steradians in a complete sphere). As with the flux and luminosity, one can also define a **specific intensity**, I_ν , which is the intensity per unit spectral bandwidth ($[I_\nu] = \text{erg s}^{-1} \text{cm}^{-2} \text{Hz}^{-1} \text{sr}^{-1}$).

The flux emerging from the surface of a star with luminosity L and radius R_* is

$$F \equiv \frac{L}{4\pi R_*^2} = \int_{\text{half sphere}} I \cos \theta d\Omega = \int_0^{2\pi} d\phi \int_0^{\pi/2} d\theta I \cos \theta \sin \theta = \pi I$$

where we have used that $d\Omega = \sin \theta d\theta d\phi$, and the fact that the integration over the solid angle Ω is only to be performed over half a sphere. Note that an observer can only measure the surface brightness of *resolved* objects; if unresolved, the observer can only measure the objects flux.

Consider a resolved object (i.e., a galaxy), whose surface brightness distribution on the sky is given by $I(\Omega)$. If the objects extents a solid angle Ω_S on the sky, its flux

is given by

$$f = \int_{\Omega_S} I(\Omega) \cos \theta \, d\Omega \simeq \int I(\Omega) \, d\Omega \equiv \langle I \rangle \Omega_S$$

where we have assumed that Ω_S is small, so that variations of $\cos \theta$ across the object can be neglected. Since both $f \propto r^{-2}$ and $\Omega_S \propto r^{-2}$, where r is the object's distance, we see that the average surface brightness $\langle I \rangle$ is independent of distance.

Energy density: the energy density, u , is a measure of the radiative energy per unit volume (i.e., $[u] = \text{erg cm}^{-3}$). If the radiation intensity as seen from some specific location in space is given by $I(\Omega)$, then the energy density at that location is

$$u = \frac{1}{c} \int I \, d\Omega \equiv \frac{4\pi}{c} J$$

where

$$J \equiv \frac{1}{4\pi} \int I \, d\Omega$$

is the **mean intensity** (i.e., average over 4π steradian). If the radiation is isotropic (i.e., the center of a star, or, to good approximation, a random location in the early Universe), then $J = I$. If the radiation intensity is due to the summed intensity from a number of individual sources, then $u = \frac{1}{c} \sum_i f_i$, where f_i is the flux due to source i .

Recall from Chapter 6 that the number density of photons emerging from a **Black Body** of temperature T is given by

$$n_\gamma(\nu, T) \, d\nu = \frac{8\pi \nu^2}{c^3} \frac{d\nu}{e^{h\nu/k_B T} - 1}$$

Hence, we have that

$$u(\nu, T) \, d\nu = n_\gamma(\nu, T) \, h\nu \, d\nu = \frac{8\pi h \nu^3}{c^3} \frac{d\nu}{e^{h\nu/k_B T} - 1}$$

Using that $u(\nu, T) = (4\pi/c) J_\nu(T)$ we have that the **mean specific intensity** from a black body [for which one typically uses the symbol $B_\nu(T)$] is given by

$$B_\nu(T) \, d\nu = \frac{2 h \nu^3}{c^2} \frac{d\nu}{e^{h\nu/k_B T} - 1}$$

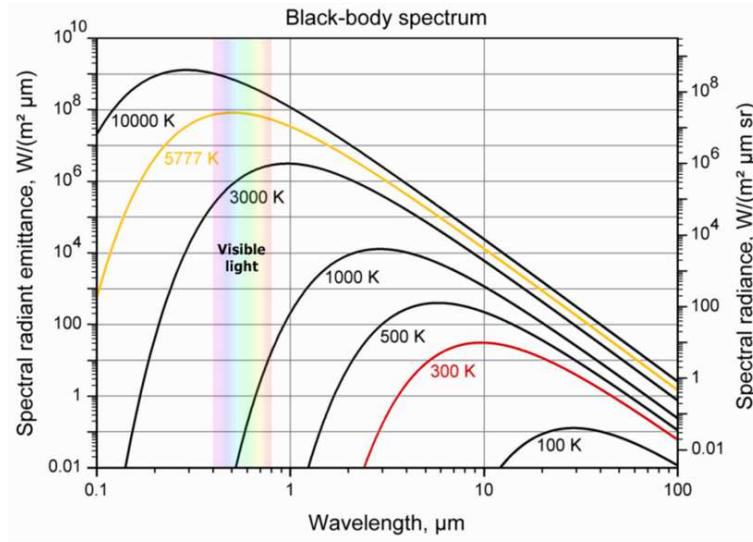


Figure 23: Various Planck curves for different temperatures, illustrating Wien's displacement law. Note how the Planck curve for a black body with the temperature of the Sun peaks at the visible wavelengths, where the sensitivity of our eyes is maximal

which is called the **Planck curve** (or 'formula'). Integrating over frequency yields the total, mean intensity emitted from the surface of a Black Body

$$J = J(T) = \int_0^\infty B_\nu(T) d\nu = \frac{\sigma_{\text{SB}}}{\pi} T^4$$

where σ_{SB} is the **Stefan-Boltzmann constant**. This implies an energy density

$$u = u(T) = \frac{4\pi}{c} J = \frac{4\sigma_{\text{SB}}}{c} T^4 \equiv a_r T^4$$

where $a_r \simeq 7.6 \times 10^{-15} \text{ erg cm}^{-3} \text{ K}^{-4}$ is called the **radiation constant** (see also Chapter 6).

Wien's Displacement Law: When the temperature of a Black Body emitter increases, the overall radiated energy increases and the peak of the radiation curve moves to shorter wavelengths. It is straightforward to show that the product of the temperature and the wavelength at which the Planck curve peaks is a constant, given by

$$\lambda_{\text{max}} T = 0.29$$

where T is the *absolute temperature*, expressed in degrees Kelvin, and $\lambda_{\max}$ is expressed in cm. This relation is called **Wien's Displacement Law**.

Stefan-Boltzmann Law: The flux emitted by a Black Body is

$$F_{\text{BB}} = \pi I(T) = \sigma_{\text{SB}} T^4$$

which is known as the **Stefan-Boltzmann law**. This law is used to define the effective temperature of an emitter.

Effective Temperature: The temperature an emitter of flux F would have if it were a Black Body; using the Stefan-Boltzmann law we have that $T_{\text{eff}} = (F/\sigma_{\text{SB}})^{1/4}$. We can also use the effective temperature to express the emitter's luminosity;

$$L = 4 \pi R^2 \sigma_{\text{SB}} T_{\text{eff}}^4$$

where R is the emitter's radius. The effective temperature is sometimes also called the **radiation temperature**, as a measure for the temperature associated with the radiation field.

Brightness Temperature: the brightness temperature, $T_{\text{B}}(\nu)$, of a source at frequency ν is defined as the temperature which, when put into the Planck formula, yields the specific intensity actually measured at that frequency. Hence, for a Black Body $T_{\text{B}}(\nu)$ is simply equal to the temperature of the Black Body. If $T_{\text{B}}(\nu)$ depends on frequency, then the emitter is not a Black Body. The brightness temperature is a frequency-dependent version of the effective, or radiation, temperature.

Wavebands: Astronomers typically measure an object's flux through some filter (waveband). The measured flux in 'band' X is, f_X , is related to the spectral flux density, f_{λ} , of the object according to

$$f_X = \int f_{\lambda} F_X(\lambda) R(\lambda) T(\lambda) d\lambda$$

Here $F_X(\lambda)$ describes the transmission of the filter that defines waveband X , $R(\lambda)$ is the transmission efficiency of the telescope + instrument, and $T(\lambda)$ describes the transmission of the atmosphere. The combined effect of F_X , R , and T is typically 'calibrated' using standard stars with known f_{λ} .

Magnitudes: For historical reasons, the flux of an astronomical object in waveband X is usually quoted in terms of **apparent magnitude**:

$$m_X = -2.5 \log \left(\frac{f_X}{f_{X,0}} \right)$$

where the flux zero-point $f_{X,0}$ has traditionally been taken as the flux in the X band of the bright star Vega. In recent years it has become more common to use ‘AB-magnitudes’, for which

$$f_{X,0} = 3.6308 \times 10^{-20} \text{ erg s}^{-1} \text{ cm}^{-2} \text{ Hz}^{-1} \int F_X(c/\nu) d\nu$$

Similarly, the luminosities of objects (in waveband X) are often quoted as an **absolute magnitude**:

$$M_X = -2.5 \log(L_X) + C_X$$

where C_X is a zero point. It is usually convenient to write L_X in units of the solar luminosity in the same band, $L_{\odot X}$, so that

$$M_X = -2.5 \log \left(\frac{L_X}{L_{\odot X}} \right) + M_{\odot X},$$

where $M_{\odot X}$ is the absolute magnitude of the Sun in the waveband in consideration. Using the relation between luminosity and flux we have that

$$m_X - M_X = 5 \log(r/r_0)$$

where r_0 is a fiducial distance at which m_X and M_X are defined to have the same value. Conventionally, r_0 is chosen to be 10 pc.

Distance modulus: the distance modulus of an object is defined as $m_X - M_X$.

CHAPTER 19

Thermal Equilibrium & Saha Equation

Most of the baryonic matter in the Universe is in a gaseous state, made up of $\sim 75\%$ Hydrogen (H), $\sim 25\%$ Helium (He) and only small amounts of other elements (called ‘metals’). Gases can be neutral, ionized, or partially ionized. The degree of ionization of any given element is specified by a Roman numeral after the element name. For example HI and HII refer to neutral and ionized hydrogen, respectively, while CI is neutral carbon, CII is singly-ionized carbon, and CIV is triply-ionized carbon. A gas that is highly (largely) ionized is called a **plasma**.

Thermodynamic Equilibrium: a system is said to be in thermodynamic equilibrium (TE) when it is in

- thermal equilibrium
- mechanical equilibrium
- radiative equilibrium
- chemical equilibrium
- statistical equilibrium

Equilibrium means a state of balance. In a state of thermodynamic equilibrium, there are no net flows of matter or of energy, no phase changes, and no unbalanced potentials (or driving forces), within the system. A system that is in thermodynamic equilibrium experiences no changes when it is isolated from its surroundings.

In a system in TE the energy in the radiation field is in equilibrium with the kinetic energy of the particles. If the system is isolated (to both matter and radiation), and in mechanical equilibrium, then over time TE will be established. For a gas in TE, the **radiation temperature**, T_R , is equal to the **kinetic temperature**, T , is equal to the **excitation temperature**, T_{ex} (see below for definitions). Since no photons are allowed to escape from a system in TE (this would correspond to energy loss, and thus violate TE), the photons that are produced in the gas (represented by T_R) therefore must be tightly coupled to the random motions of the particles (represented by T). This coupling implies that, for bound states, the excitation and de-excitation of the atoms and ions *must* be dominated by **collisions**. In other words, the collision timescales must be shorter than the timescales associated with photon interactions

or **spontaneous de-excitations**. If that were not the case, T could not remain equal to T_{R} .

Local Thermodynamic Equilibrium (LTE): True TE is rare (in almost all cases energy does escape the system in the form of radiation, i.e., the system cools), and often temperature gradients are present. A good, albeit imperfect, example of TE is the Universe as a whole prior to decoupling. Although true TE is rare, in many systems (stars, gaseous spheres, ISM), we may apply local TE (LTE), which implies that the gas is in TE, but only *locally*. In a system in LTE, there will typically be gradients in temperature, density, pressure, etc, but they are sufficiently small over the mean-free path of a gas particle. For stellar interiors, the fact that radiation is ‘locally trapped’ explains why it takes so long for photons to diffuse from the center (where they are created in nuclear reactions) to the surface (where they are emitted into space). In the case of the Sun, this timescale is of the order of 200,000 years.

Thermal Equilibrium: For a system to be in thermal equilibrium with itself, it must not contain temperature gradients. For two systems to be in thermal equilibrium with each other requires an actual or implied thermal connection between them, through a path that is permeable only to heat, and that no **net** energy is transferred through that path.

For a gas in **thermal equilibrium** at some uniform temperature, T , and uniform number density, n , the number density of particles with speeds between v and $v + dv$ is given by the **Maxwell-Boltzmann** (aka ‘Maxwellian’) velocity distribution:

$$n(v) dv = n \left(\frac{m}{2\pi k_{\text{B}} T} \right)^{3/2} \exp \left(-\frac{m v^2}{2 k_{\text{B}} T} \right) 4\pi v^2 dv$$

where m is the mass of a gas particle. The temperature T is called the **kinetic temperature**, and is related to the mean-square particle speed, $\langle v^2 \rangle$, according to

$$\frac{1}{2} m \langle v^2 \rangle = \frac{3}{2} k_{\text{B}} T$$

The *most probable speed* of the Maxwell-Boltzmann distribution is $v_{\text{mp}} = \sqrt{2k_{\text{B}}T/m}$, while the *mean speed* is $v_{\text{mean}} = \sqrt{8k_{\text{B}}T/m}$.

Setting up a Maxwellian velocity distribution requires many **elastic** collisions. In the limit where most collisions are elastic, the system will typically very quickly equilibrate to thermal equilibrium. However, depending on the temperature of the gas, collisions can also be **inelastic**: examples of the latter are **collisional excitations**, in which part of the kinetic energy of the particle is used to excite its target to an excited state (i.e., the kinetic energy is now temporarily stored as a potential energy). If the **de-excitation** is **collisional**, the energy is given back to the kinetic energy of the gas (i.e., no photon is emitted). However, if the **de-excitation** is **spontaneous** or via **stimulated emission**, a photon is emitted. If the gas is optically thin to the emitted photon, the energy will escape the gas. The net outcome of the collisional excitation is then one of dissipation, i.e., cooling (kinetic energy of the gas is being radiated away). In the optically thick limit, the photon will be absorbed by another atom (or ion). If the system is in (L)TE, the **radiation temperature** of these photons being emitted and absorbed is equal to the **kinetic temperature** of the gas. Although in LTE a subset of the collisions are inelastic, this subset is typically small, and one may still use the Maxwell-Boltzmann distribution to characterize the velocities of the gas particles.

Mechanical Equilibrium: A system is said to be in mechanical equilibrium if (i) the vector sum of all *external* and *internal* forces is zero and (ii) the sum of the moments of all *external* and *internal* forces about any line is zero. The internal forces arise from pressure gradients. If a system is out of mechanical equilibrium, it will try to re-establish mechanical equilibrium, the characteristic time-scale of which is the **sound-crossing time**.

Radiative Equilibrium: A system is said to be in radiative equilibrium if any two randomly selected subsystems of it exchange *by radiation* equal amounts of heat with each other.

Chemical Equilibrium: In a chemical reaction, chemical equilibrium is the state in which both reactants and products are present at concentrations which have no further tendency to change with time. Usually, this state results when the forward reaction proceeds at the same rate as the reverse reaction. The reaction rates of the forward and reverse reactions are generally not zero but, being equal, there are no net changes in the concentrations of the reactant and product.

Statistical Equilibrium: A system is said to be in statistical equilibrium if the level populations of its constituent atoms and ions do not change with time (i.e., if the transition rate into any given level equals the rate out).

For a system in statistical equilibrium, the population of states is described by the **Boltzmann law** for level populations:

$$\frac{N_j}{N_i} = \frac{g_j}{g_i} \exp\left(-\frac{h \nu_{ij}}{k_B T}\right)$$

Here N_i is the number of atoms in which electrons are in energy level i (here i reflects the principal quantum number, with $i = 1$ referring to the *ground state*), ν_{ij} is the frequency corresponding to the energy difference $\Delta E_{ij} = h \nu_{ij}$ of the energy levels, h is the Planck constant, and g_i is the **statistical weight** of population i .

Statistical weight: the statistical weight (aka ‘degeneracy’) indicates the number of states at a given principal quantum number, n . In quantum mechanics, a total of four quantum numbers is needed to describe the state of an electron: the principle quantum number, n , the orbital angular momentum quantum number, l , the magnetic quantum number m_l , and the electron spin quantum number, m_s . For hydrogen l can take on the values $0, 1, \dots, n-1$, the quantum number m_l can take on the values $-l, -l+1, -1, 0, 1, \dots, l$, and m_s can take on the values $+1/2$ (‘up’) and $-1/2$ (‘down’). Hence, $g_n = 2n^2$.

Excitation Temperature: The above **Boltzmann law** defines the excitation temperature, T_{ex} , as the temperature which, when put into the Boltzmann law for level populations, results in the observed ratio of N_j/N_i . For a gas in (local) TE, all levels in an atom can be described by the same T_{ex} , which is also equal to the **kinetic temperature**, T , and the radiation temperature, T_{R} . Under non-LTE conditions, each pair of energy levels can have a different T_{ex} .

Radiation Temperature: the radiation temperature, or T_{R} , is a temperature that specifies the energy density in the radiation field, u_γ , according to $u_\gamma = a_r T_{\text{R}}^4$. In TE $T_{\text{R}} = T$, while for a Black Body, T_{R} is the temperature that appears in the **Planck function** that describes the specific intensity of the radiation field (see Appendix H for more details).

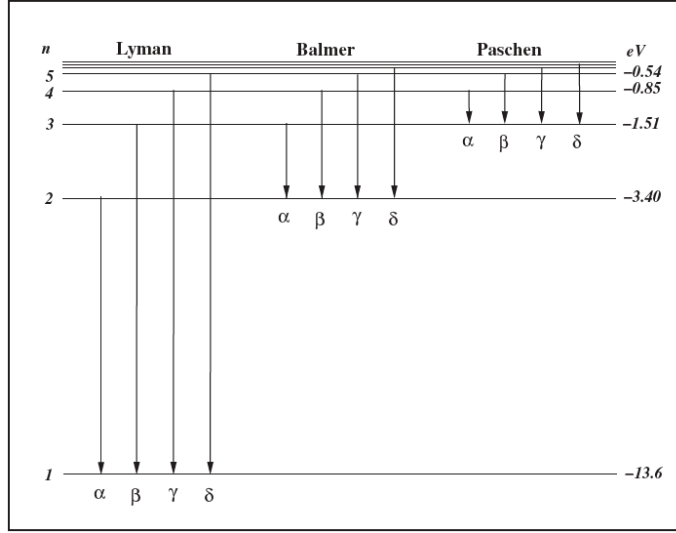


Figure 24: *Illustration of the energy levels of the hydrogen atom and some of the most important transitions.*

Partition Function: Using the above **Boltzmann law**, it is straightforward to show that

$$\frac{N_n}{N} = \frac{g_n}{U} \exp\left(-\frac{\Delta E_n}{k_B T}\right)$$

where $N = \sum N_i$ is the total number of atoms (of all energy levels) and ΔE_n is the energy difference between state n and the ground-state, and

$$U = \sum g_i e^{-\Delta E_i/k_B T}$$

is called the **partition function**. Note that at low temperatures, below that needed to put a significant fraction of atoms in the first excited state, the partition function becomes equal to the statistical weight of the ground state: $U = g_1 \exp -0/k_B T = g_1 = 2$. After all, the exponential factors for the excited states are all extremely small. (Note: the summation is over all principal quantum numbers up to some maximum $n_{\max}$, which is required to prevent divergence; see **Irwin 2007** or **Rybicki & Lightmann 1979** for details).

Saha equation: the Saha equation expresses the ratios of atoms/ions in different ionization states. In particular, the number of atoms/ions in the $(K+1)^{\text{th}}$ ionization

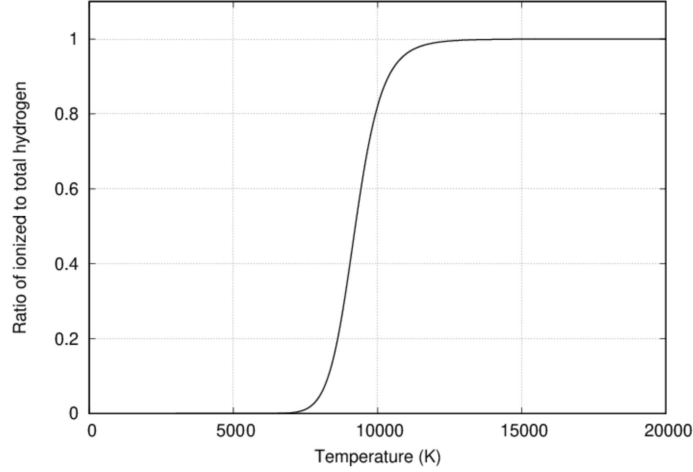


Figure 25: *The ionization fraction of hydrogen as a function of temperature, computed using the Saha equation.*

state versus those in the K^{th} ionization state is given by

$$\frac{N_{K+1}}{N_K} = \frac{2}{n_e} \frac{U_{K+1}}{U_K} \left(\frac{2\pi m_e k_B T}{h^2} \right)^{3/2} \exp \left(-\frac{\chi_K}{k_B T} \right)$$

where U_n is the **partition function** of the n^{th} ionization state, n_e is the electron number density, and χ_K is the energy required to remove an electron from the ground state of the K^{th} ionization state. Note that unlike the Boltzmann equation, the Saha equation for the ionization fractions has a dependence on electron density. This reflects that if there is a higher density of free electrons, there is a greater probability that an electron will recombine with the ion, lowering the ionization state of the gas.

Hydrogen Since Hydrogen is the most common element in the Universe, it is important to have some understanding of its structure. Fig. 18 shows the energy levels of a hydrogen atom and some of its most important transitions. The Balmer transitions have wavelengths that fall in the optical, and are therefore well known to (optical) astronomers. The Lyman lines typically fall in the UV, and can only be observed from space (or for high-redshift objects, for which the rest-frame UV is redshifted into the optical).

Using the **Saha equation**, we can compute the **ionization fraction** for hydrogen as a function of temperature and electron density. The ionization fraction is $x =$

$N_{\text{HII}}/[N_{\text{HI}} + N_{\text{HII}}]$, which can be computed using the Saha equation:

$$\frac{N_{\text{HII}}}{N_{\text{HI}}} = 2.41 \times 10^{15} \frac{T^{3/2}}{n_e} \exp\left(-\frac{1.58 \times 10^5}{T}\right)$$

where we have used that $\chi_{\text{HI}} = 13.6\text{eV}$, $U_{\text{HI}} = 2$ and $U_{\text{HII}} = 1$ (i.e., the ionized hydrogen atom is just a free proton and only exists in a single state). Figure . 24 shows the ionization fraction x as function of temperature. Note that the transition from almost neutral to almost completely ionized is extremely rapid! We can also compute, using the **Boltzmann law**, the ratio of hydrogen atoms in the first excited state compared to those in the ground state. The latter shows that the temperature must exceed 30,000K in order for there to be an appreciable (10 percent) number of hydrogen atoms in the first excited state. However, at such high temperatures, one typically has that most of the hydrogen will be ionized (unless the electron density is unrealistically high). This somewhat unintuitive result arises from the fact that there are many more possible states available for a free electron than for a bound electron in the first excited state. In conclusion, *neutral hydrogen in LTE will have virtually all its atoms in the ground state.*

If densities are sufficiently low, which is typically the case in the ISM, the collisional excitation rate (which scales with n_e^2) is lower than the spontaneous de-excitation rate. If that is the case, the hydrogen gas is no longer in LTE, and once again, virtually all neutral hydrogen will find itself in the ground state. *Neutral hydrogen in the ISM is in the ground-state and is typically NOT in LTE.* One important consequence of the fact that neutral hydrogen is basically always observed in the ground state, is that observations of hydrogen emission lines (Lyman, Balmer, Paschen, etc) indicates that the hydrogen *must* be ionized; the lines arise from recombinations, and are therefore called **recombination lines**. Note that Balmer lines are often observed in absorption (for example, Balmer lines are evident in a spectrum of the Sun). This implies that their must be hydrogen present in the first excited state, which seems at odds with the conclusions reached above. A small fraction of excited hydrogen atoms, though, can still produce strong absorption lines, simply because hydrogen is so abundant.

21cm line emission: The ground state of hydrogen is split into two hyperfine states due to the two possible orientations of the proton and electron spins: The state in which the spins of proton and electron are aligned has slightly higher energy than the one in which they are anti-aligned. The energy difference between these

two hyperfine states corresponds to a photon with a wavelength of 21cm (which falls in the radio). The excitation temperature of this **spin-flip** transition is called the **spin temperature**. Since, for typical interstellar medium conditions, the spin-flip transition is *collisionally induced* (i.e., the rate for spontaneous de-excitation is extremely low), the spin-temperature is typically equal to the kinetic temperature of the hydrogen gas. The 21cm line is an important emission line to probe the distribution (and temperature) of neutral hydrogen gas in the Universe.

CHAPTER 20

The Interaction of Light with Matter: I - Scattering

To understand radiative processes, and the interaction of photons with matter, it is important to realize that *all* photon emission mechanisms arise from **accelerating electrical charge**.

The interactions of light with matter can be split in two categories:

- **scattering** (photon + matter $\rightarrow$ photon + matter)
- **absorption** (photon + matter $\rightarrow$ matter)

We first discuss **scattering**, which gives rise to a number of astrophysical phenomena:

- reflection nebulae (similar to looking at street-light through fog)
- light echos
- polarization
- Ly- α forest in quasar spectra

The **scattering cross-section**, σ_s , is a hypothetical area ($[\sigma_s] = \text{cm}^2$) which describes the likelihood of a photon being scattered by a target (typically an electron or atom). In general, the scattering cross-section is different from the geometrical cross-section of the particle, and it depends upon the frequency of the photon, and on the details of the interaction (see below).

Scattering interactions are categorized as either **elastic (coherent)**, where the photon energy is unchanged by the scattering event, or **inelastic (incoherent)**, where the photon energy changes.

Elastic scattering comes in three forms:

- **Thomson scattering** $\gamma + e \rightarrow \gamma + e$
- **Resonant scattering** $\gamma + X \rightarrow X^+ \rightarrow \gamma + X$
- **Rayleigh scattering** $\gamma + X \rightarrow \gamma + X$

Here γ indicates a photon, e a free electron, X an atom or ion, and X^+ an excited state of X .

Inelastic scattering comes in two forms:

- **Compton scattering** $\gamma + e \rightarrow \gamma' + e'$
- **Fluorescence** $\gamma + X \rightarrow X^{++} \rightarrow \gamma' + X^+ \rightarrow \gamma' + \gamma'' + X$

Here accents indicate that the particle has a different energy (i.e., γ' is a photon with a different energy than γ), and X^{++} indicates a higher-excited state of X than X^+ .

In what follows we discuss each of these five processes in more detail.

Thomson scattering: is the elastic (coherent) scattering of electromagnetic radiation by a free charged particle, as described by classical electromagnetism. It is the low-energy limit of **Compton scattering** in which the particle kinetic energy and photon frequency are the same before and after the scattering. In Thomson scattering the electric field of the incident wave (photon) accelerates the charged particle, causing it, in turn, to emit radiation at the same frequency as the incident wave, and thus the wave is scattered. The particle will move in the direction of the oscillating electric field, resulting in **electromagnetic dipole radiation** that appears **polarized** unless viewed in the forward or backward scattered directions (see Fig. 26).

The cross-section for Thomson scattering is the **Thomson cross section**:

$$\sigma_s = \sigma_T = \frac{8\pi}{3} r_e^2 = \frac{8\pi e^4}{3m_e^2 c^4} \simeq 6.65 \times 10^{-25} \text{cm}^2$$

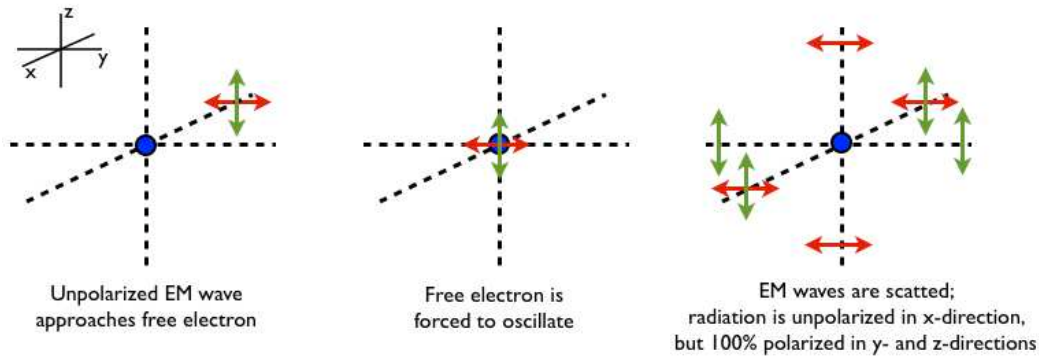


Figure 26: *Illustration of how Thomson scattering causes polarization in the directions perpendicular to that of the incoming EM radiation. The incoming EM wave causes the electron to oscillate in the direction of the oscillation of the $\vec{E}$ -field. This acceleration of the electrical charge results in the emission of dipolar EM radiation.*

Note that this cross section is independent of wavelength!

In the quantum mechanical view of radiation, electromagnetic waves are made up of photons which carry both energy ($h\nu$) and momentum ($h\nu/c$). This implies that during scattering the photon exchanges momentum with the electron, causing the latter to recoil. This **recoil** is negligible, until the energy of the incident photon becomes comparable to the rest-mass energy of the electron, in which case Thomson scattering becomes Compton scattering.

Compton scattering: is an inelastic scattering of a photon by a free charged particle, usually an electron. It results in a decrease of the photon's energy/momentum (increase in wavelength), called the Compton effect. Part of the energy/momentum of the photon is transferred to the scattering electron ('recoil'). In the case of scattering off of *electrons at rest*, the Compton effect is only important for high-energy photons with $E_\gamma > m_e c^2 \sim 0.511 \text{ MeV}$ (X-ray and/or gamma ray photons).

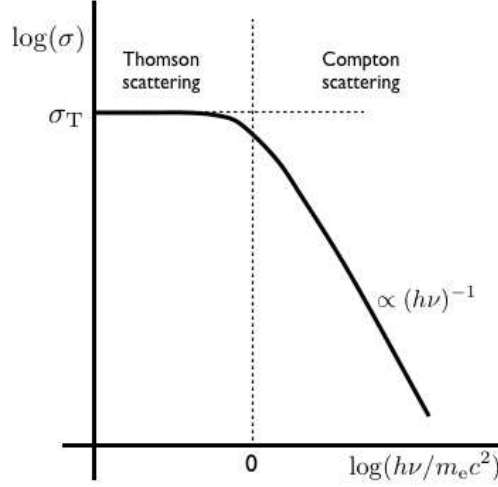


Figure 27: *The Klein-Nishina cross section for Compton scattering. As long as $h\nu \ll m_e c^2$ one is in the Thomson scattering regime, and $\sigma_s = \sigma_T$. However, once the photon energy becomes comparable to the rest-mass energy of the electron, Compton scattering takes over, and the cross-section (now called the Klein-Nishina cross-section), starts to drop as ν^{-1} .*

Because of the recoil effect, the energy of the outgoing photon is

$$E'_\gamma = \frac{E_\gamma}{1 + \frac{E_\gamma}{m_e c^2} (1 - \cos \theta)}$$

where θ is the angle between in incident and outgoing photon. This can also be written as

$$\lambda' - \lambda = \lambda_C (1 - \cos \theta)$$

which expresses that Compton scattering increases the wavelength of the photon by of order the **Compton wavelength** $\lambda_C = h/(m_e c) \sim 2.43 \times 10^{-10} \text{cm}$. If $\lambda \gg \lambda_C$ such a shift is negligible, and we are in the regime that is well described by **Thomson scattering**.

Compton scattering is a quantum-mechanical process. The quantum aspect also influences the actual cross section, which changes from the Thomson cross section, σ_T , at the low-frequency end, to the **Klein-Nishina cross section**, $\sigma_{\text{KN}}(\nu)$, for $h\nu > m_e c^2$ (see Fig. 27). Note how scattering becomes less efficient for more energetic photons.

So far we have considered the scattering of photons off of *electrons at rest*. A more realistic treatment takes into account that electrons are also moving, and may do so relativistically. This adds the possibility of the electron giving some of its kinetic energy to the photon, which results in **Inverse Compton (IC) scattering**.

Whether the photon loses (Compton scattering) or gains (IC scattering) energy depends on the energies of the photon and electron. Without derivation, the average energy change of the photon *per Compton scattering* against electrons of temperature $T_e = m_e \langle v^2 \rangle / (3 k_B)$ is

$$\boxed{\left\langle \frac{\Delta E_\gamma}{E_\gamma} \right\rangle = \frac{4 k_B T_e - h \nu}{m_e c^2}}$$

Hence, we have that

- $h \nu > 4 k_B T_e$: Compton effect; photon loses energy to electron
- $h \nu = 4 k_B T_e$: No energy exchange
- $h \nu < 4 k_B T_e$: Inverse Compton effect; electron loses energy to photon

As an example, consider **ultra-relativistic electrons** with $4 k_B T_e \gg h \nu$. In that case it can be shown that Inverse Compton (IC) scattering causes the photons to increase their frequency according to $\nu_{\text{out}} \approx \gamma^2 \nu_{\text{in}}$, where

$$\gamma = \frac{1}{\sqrt{1 - \frac{v^2}{c^2}}}$$

is the **Lorentz factor**. Hence, for ultra-relativistic electrons, which have a large Lorentz factor, the frequency boost of a single IC scattering event can be enormous. It is believed that this process, upscattering of low energy photons by the IC effect, is at work in Active Galactic Nuclei.

Another astrophysical example of IC scattering is the **Sunyaev-Zel'dovic (SZ) effect** in clusters; the hot (but non-relativistic) electrons of the intra-cluster gas (with a typical electron temperature of $T_e \sim 10^8$ K) upscatter Cosmic Microwave Background (CMB) photons by a small, but non-negligible amount. The result is a **comptonization** of the energy spectrum of the photons; while Compton scattering maintains photon numbers, it increases their energies, so that they no longer can be fit by a Planck curve. The strength of this Comptonization (typically expressed in

terms of the Compton- y parameter) is measure for the electron pressure $P_e \propto n_e T_e$ along the line-of-sight through the cluster. Observations of the SZ effect provide a nearly redshift-independent means of detecting galaxy clusters.

Resonant scattering: Resonant scattering, also known as **line scattering** or **bound-bound scattering** is the scattering of photons off electrons bound to nuclei in atoms or ions. Before we present the quantum mechanical view of this process, it is useful to consider the classical one, in which the electron is viewed as being bound to the nucleus via a spring with a **natural, angular frequency**, $\omega_0 = 2\pi\nu_0$. If the electron is perturbed, it will oscillate at this natural frequency, which will result in the emission of photons of energy $E_\gamma = h\nu_0$. This in turn implies energy loss; hence, the bound electron is an example of a **damped, harmonic oscillator**. The classical *damping constant* is given by $\Gamma_{\text{cl}} = \omega_0^2 \tau_e$, where $\tau_e = 2e^2/(3m_e c^2) \sim r_e/c \sim 6.3 \times 10^{-24}\text{s}$. This damping, and the corresponding emission of EM radiation, is the classical analog of **spontaneous emission**.

Now consider the case of an EM wave of angular frequency, ω , interacting with the atom/ion. The result is a **forced, damped, harmonic oscillator**, whose effective cross section is given by

$$\sigma_s(\omega) = \sigma_T \frac{\omega^4}{(\omega^2 - \omega_0^2)^2 + (\omega_0^3 \tau_e)^2}$$

(see Rybicki & Lightmann 1979 for a derivation). We can distinguish three regimes:

$\boxed{\omega \gg \omega_0}$ In this case $\sigma_s(\omega) = \sigma_T$ and we are in the regime of regular Thomson scattering. The oscillator responds to the high-frequency forcing by adopting the forced frequency; hence, the system behaves as if the electron is free.

$\boxed{\omega \simeq \omega_0}$ In this case

$$\sigma_s(\omega) \simeq \frac{\sigma_T}{2\tau_e} \frac{(\Gamma_{\text{cl}}/2)}{(\omega - \omega_0)^2 + (\Gamma_{\text{cl}}/2)^2}$$

which corresponds to **resonant scattering**, in which the cross section is hugely boosted wrt the Thomson case. NOTE: for resonant scattering to be important, it is crucial that *spontaneous* de-excitation occurs before *collisional* excitation or de-excitation (otherwise the photon energy is lost, and we are in the realm of absorption, rather than scattering). Typically, this requires sufficiently low densities.

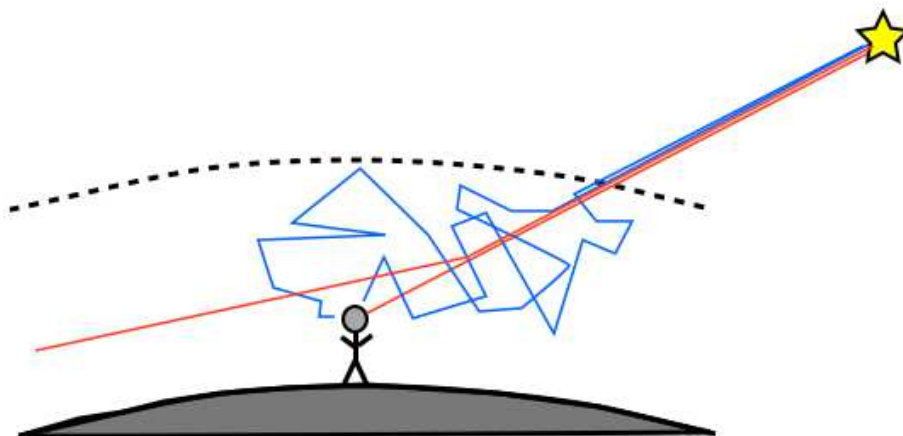


Figure 28: *Illustration of how Rayleigh scattering causes the sky to be blue. Because of its strong (λ^{-4}) wave-length dependence, blue light is much more scattered than red light. This causes the Sun light to appear redder than it really is, an effect that strengthens when the path length through the atmosphere is larger (i.e., at sunrise and sunset). The blue light is typically scattered multiple times before hitting the observer, so that it appears to come from random directions on the sky.*

$\omega \ll \omega_0$ In this case

$$\sigma_s(\omega) \simeq \sigma_T \left(\frac{\omega}{\omega_0} \right)^4$$

which corresponds to **Rayleigh scattering**, which is characterized by a strong wave-length dependence for the effective cross section of the form $\sigma_s \propto \sigma_T \lambda^{-4}$.

Rayleigh scattering results from the electric polarizability of the particles. The oscillating electric field of a light wave acts on the charges within a particle, causing them to move at the same frequency (recall, the forcing frequency in this case is much smaller than the natural frequency). The particle therefore becomes a small radiating dipole whose radiation we see as scattered light.

Rayleigh scattering, and its strong wavelength dependence of σ_s , is responsible for the fact that the sky appears blue during the day, and for the fact that sunsets turn the sky red (see Fig. 28).

We now turn our attention to a quantum-mechanical view of **resonant scattering**. The main difference between the classical view (above) and the quantum view (below), is that in the latter there is not one, but many ‘natural frequencies’, ν_{ij} , corresponding to all the possible energy-level-transitions $\Delta E_{ij} = h\nu_{ij}$ that correspond to the atom/ion in question.

Oscillator strength: With each transition corresponds an oscillator strength, f_{ij} , which is a dimensionless quantity that expresses the ‘strength’ of the $i \leftrightarrow j$ transition. It expresses the quantum mechanical probability that transition $i \rightarrow j$ occurs under the incidence of a ν_{ij} photon given the quantum-mechanical selection rules, which state the degree to which a certain transition between degrees is allowed. You can think of f_{ij} as being proportional to the probability that the incidence of a ν_{ij} photon results in the corresponding electronic transition.

In the quantum-mechanical view, the three regimes of bound-bound scattering have effective cross sections:

$$\begin{array}{lll} \sigma_s = \sigma_T & \nu \gg \nu_{ij} & \textbf{Thomson scattering} \\ \sigma_s(\nu) = \frac{\pi e^2}{m_e c} f_{ij} \phi_{\mathcal{L}}(\nu) & \nu \simeq \nu_{ij} & \textbf{Resonant scattering} \\ \sigma_s(\nu) = \sigma_T f_{ij} \left(\frac{\nu}{\nu_{ij}} \right)^4 & \nu \ll \nu_{ij} & \textbf{Rayleigh scattering} \end{array}$$

Here $\phi_{\mathcal{L}}(\nu)$ is the **Lorentz profile**, which describes the **natural line broadening** associated with the transition in question. The non-zero width of this Lorentz profile implies that resonant scattering is not perfectly coherent; typically the energy of the outgoing photon will be slightly different from that of the incident photon. The probability distribution for this energy shift is described by $\phi_{\mathcal{L}}(\nu)$, and originates from the **Heisenberg Uncertainty Principle**, according to which $\Delta E \Delta t \geq \hbar/2$; hence, the uncertainty related to the time it takes for the electron to spontaneously de-excite results in a related ‘uncertainty’ in energy.

Fig. 29 shows the frequency dependence of a (quantum-mechanical) atom/ion. It shows the **Rayleigh regime** at small ν , the **resonant scattering** peaks at a few transition frequencies, and the **Thomson regime** at large ν . Note that the height of the various peaks are set by their respective **oscillator strengths**.

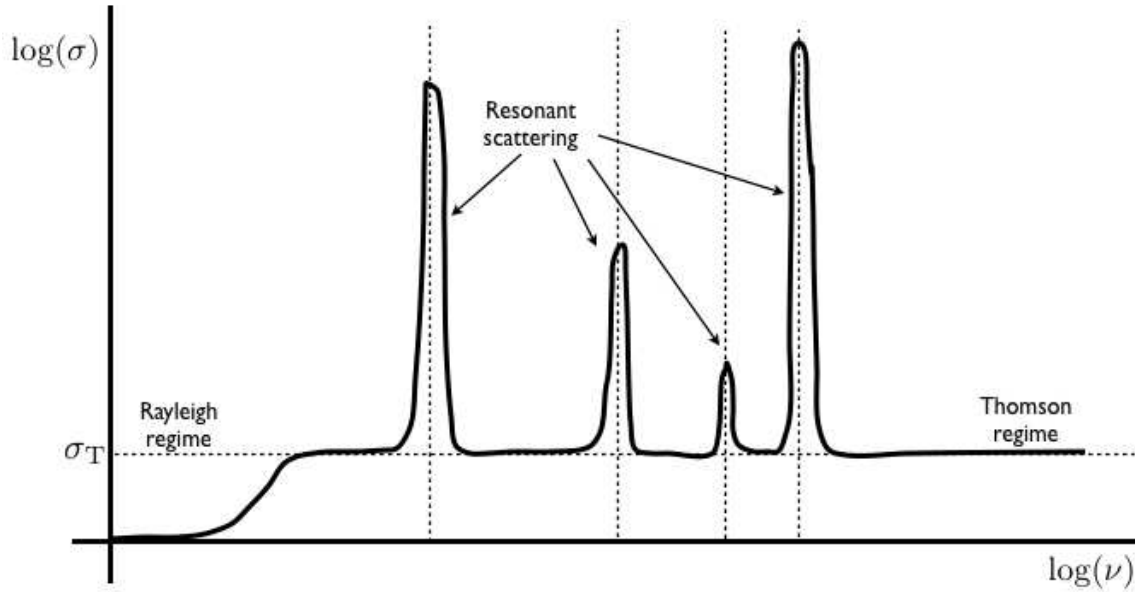


Figure 29: *Illustration of the scattering cross section of an atom or ion with at least one bound electron. At high (low) frequency, scattering is in the Thomson (Rayleigh) regime; at specific, intermediate frequencies, set by the transition energies of the atom/ion, resonant scattering dominates; the profiles are Lorentz profiles, and reflect the natural line broadening. The relative heights of the peaks are set by their oscillator strengths. NOTE: figure is not to scale; typically the cross section for resonant scattering is orders of magnitude larger than the Thomson cross section.*

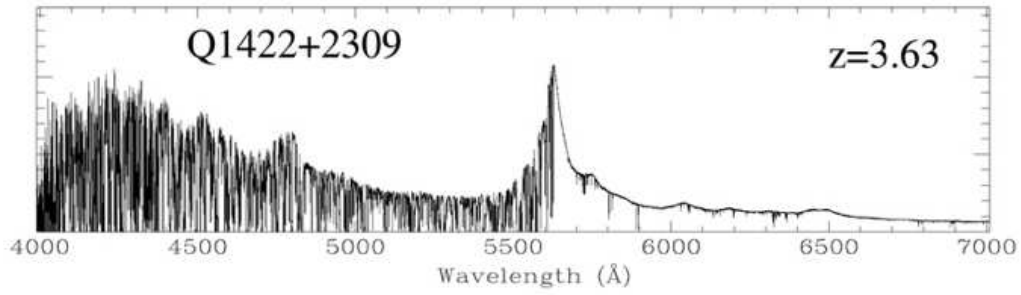


Figure 30: *Example of a quasar spectrum revealing the Ly- α forest due to resonant scattering of Ly- α photons by neutral hydrogen along the line-of-sight from quasar to observer.*

An important, astrophysical example of resonant scattering is the **Ly- α forest** in the spectra of (high-redshift) quasars. Redward of the quasar's Ly- α emission line one typically observes a 'forest' of 'absorption lines', called the Ly- α forest (see Fig. 30). These arise from resonant scattering in the Ly- α line of neutral hydrogen in gas clouds along the line-of-sight between the quasar and observer. NOTE: although these are called 'absorption lines' they really are a manifestation of (resonant) scattering.

Fluorescence: fluorescence is an inelastic (incoherent) scattering mechanism, in which a photon excites an electron by at least two energy states, and the spontaneous de-excitation occurs to one or more of the intermediate energy levels. Consequently, the photon that is 'scattered' (i.e., absorbed and re-emitted) has changed its energy.

CHAPTER 21

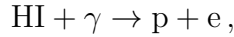
The Interaction of Light with Matter: II - Absorption

Absorption: The absorption of photons can have three effects:

- **heating of the absorbing medium** (heating of dust grains, or excitation of gas followed by collisional de-excitation)
- **acceleration of absorbing medium** (radiation pressure)
- **change of state of absorbing medium** (ionization, sublimation or dissociation)

Note that ionization (transition from neutral to ionized), sublimation (transition from solid to gas) and dissociation (transition from molecular to atomic) can also occur as a consequence of particle collisions. Therefore one often uses terms such as *photo-ionization* and *collisional ionization* to distinguish between these.

Photoionization: Photoionization is the process in which an atom is ionized by the absorption of a photon. For hydrogen, this is



where HI denotes a neutral hydrogen atom. Using that the rate, Γ , of an interaction is always given by $\Gamma = t_{\text{coll}}^{-1} = n\sigma v$, the photoionization rate, $\Gamma_{\gamma, \text{H}}$, is proportional to the number density of ionizing photons and to the photoionization cross section, $\sigma_{\text{pi}}(\nu)$, according to:

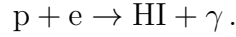
$$\Gamma_{\gamma, \text{H}} = \int_{\nu_{\text{t}}}^{\infty} c \sigma_{\text{pi}}(\nu) n_{\gamma}(\nu) d\nu$$

where ν_{t} is the threshold frequency for ionization (corresponding to 13.6eV in the case of hydrogen). $n_{\gamma}(\nu)d\nu$ is the number density of photons with frequencies in the range ν to $\nu + d\nu$, and is related to the energy flux of the radiation field, $J(\nu)$, by

$$n_\gamma(\nu) = \frac{4 \pi J(\nu)}{c h \nu}.$$

The photoionization cross sections can be obtained from quantum electrodynamics by calculating the bound-free transition probability of an atom in a radiation field (see e.g., Rybicki & Lightman 1979).

Recombination: Recombination is the process by which an ion recombines with an electron. For hydrogen ions (i.e. protons), the process is



For hydrogen (or a hydrogenic ion, i.e., an ion with a single electron), the **recombination cross section** to form an atom (or ion) at level n , $\sigma_{\text{rec}}(v, n)$, is related to the corresponding **photoionization cross section** by the **Milne relation**:

$$\sigma_{\text{rec}}(v, n) = \frac{g_n}{g_{n+1}} \left(\frac{h \nu}{m_e c v} \right)^2 \sigma_{\text{pi}}(\nu, n),$$

where $g_n = 2n^2$ is the statistical weight of energy level n and ν and v are related by $m_e v^2/2 = h(\nu - \nu_n)$, with $h\nu_n$ the threshold energy required to ionize an atom whose electron sits in energy state n . The **recombination coefficient** for a given level n is the product of the capture cross section and velocity, $\sigma_{\text{rec}}(v, n) v$, averaged over the velocity distribution $f(v)$. For an optically thin gas where all photons produced by recombination can escape without being absorbed, the total recombination coefficient is the sum over all n :

$$\alpha_{\text{A}} = \sum_{n=1}^{\infty} \alpha_n = \sum_{n=1}^{\infty} \int \sigma_{\text{rec}}(v, n) v f(v) dv$$

This is called the **Case A recombination coefficient**, to distinguish it from the **Case B** recombination in an optically thick gas. In Case B, recombinations to the ground level generate ionizing photons that are absorbed by the gas, so that they do not contribute to the overall ionization state of the gas. It is easy to see that the **Case B recombination coefficient** is $\alpha_{\text{B}} = \alpha_{\text{A}} - \alpha_1$.

Strömgren sphere: A sphere of ionized hydrogen (H II) around an ionizing source (e.g., AGN, O or B star, etc.). Ionization of hydrogen (from the ground state) requires a photon energy of at least 13.6eV, which implies UV photons. In a (partially) ionized medium, electrons and nuclei recombine to produce neutral atoms. The region around an ionizing source will ultimately establish **ionization equilibrium** in which the number of **ionizations** is equal to the number of **recombinations**.

Consider an ionizing source in a uniform medium of pure hydrogen. Let $\dot{N}_{\text{ion}}$ be the number of ionizing photons produced per second. The corresponding recombination rate is given by

$$\dot{N}_{\text{rec}} = n_e n_p \alpha_{\text{rec}} V = n_e^2 \alpha_B \frac{4}{3} \pi R_s^3$$

where we have used that, for a pure hydrogen gas, $n_e = n_p$, and R_s is the radius of the **Strömgren sphere** (i.e., the radius of the sphere that is going to be ionized), which can be written as

$$R_s = \left(\frac{3 \dot{N}_{\text{ion}}}{4 \pi \alpha_B n_e^2} \right)^{1/3}$$

Using that the luminosity of the ionizing source, L_* , is related to its surface intensity, I_* , according to

$$L_* = 4 \pi R_*^2 F_* = 4 \pi^2 R_*^2 I_*$$

where R_* is the radius of the ionizing source (i.e., an O-star) and we have used that $F_* = \pi I_*$ (see Chapter 18). Hence, we have that

$$\dot{N}_{\text{ion}} = 4 \pi^2 R_*^2 \int_{\nu_t}^{\infty} \frac{B_{\nu}(T)}{h \nu} d\nu = \frac{\pi L_*}{\sigma_{\text{SB}} T_{\text{eff}}^4} \int_{\nu_t}^{\infty} \frac{B_{\nu}(T)}{h \nu} d\nu$$

where we have assumed that the ionizing source is a Black Body of temperature T , and, in the second part, that $L_* = 4 \pi R_*^2 \sigma_{\text{SB}} T_{\text{eff}}^4$.

Thus, by measuring the luminosity and effective temperature of a star, and the radius of its Strömgren sphere, one can infer the (electron) density of its surroundings.

CHAPTER 22

The Interaction of Light with Matter: III - Extinction

As we have seen, there are numerous processes by which a photon can interact with matter. It is useful to define the **mean-free path**, l , for a photon and the related **opacity** and **optical depth**.

Opacity: a measure for the impenetrability to electro-magnetic radiation due to the combined effect of scattering and absorption. If the opacity is caused by *dust* we call it **extinction**.

Optical Depth: the dimensionless parameter, τ_ν , describing the opacity/extinction at frequency ν . In particular, the infinitesimal increase in optical depth along a line of sight, $d\tau_\nu$, is related to the infinitesimal path length dl according to

$$d\tau_\nu = \sigma_\nu n dl = \kappa_\nu \rho dl = \alpha_\nu dl$$

Here σ_ν is the **effective cross section** ($[\sigma_\nu] = \text{cm}^2$), κ_ν is the **mass absorption coefficient** ($[\kappa_\nu] = \text{cm}^2 \text{g}^{-1}$), α_ν is the **absorption coefficient** ($[\alpha_\nu] = \text{cm}^{-1}$), and n and ρ are the number and mass densities, respectively. The optical depth to a source at distance d is therefore

$$\tau_\nu = \int_0^d d\tau_\nu = \int_0^d \kappa_\nu(l) \rho(l) dl$$

The ISM/IGM between source and observer is said to be **optically thick (thin)** if $\tau_\nu > 1$ ($\tau_\nu < 1$).

Opacity/extinction reduces the intensity of a source according to

$$I_{\nu,\text{obs}} = I_{\nu,0} e^{-\tau_\nu}$$

where $I_{\nu,0}$ is the unextincted intensity (i.e., for $\tau_\nu = 0$).

Rosseland Mean Opacities: In the case of stars opacity is crucially important for understanding stellar structure. Opacities within stars are typically expressed in terms of the **Rosseland mean opacities**, $\bar{\kappa}$, which is a weighted average of κ_ν over frequency. Typically, one finds that $\bar{\kappa} \propto \rho T^{-3.5}$, which is known as **Kramer's opacity law**, and is a consequence of the fact that the opacity is dominated by bound-free and/or free-free absorption. A larger opacity implies stronger radiation pressure, which gives rise to the concept of the Eddington luminosity.

Eddington Luminosity: the maximum luminosity a star (or, more general, emitter) can achieve before the star's radiation pressure starts to exceed the force of gravity.

Consider an outer layer of a star with a thickness l such that $\tau_\nu \simeq \kappa_\nu \rho l = 1$. Then, a photon passing this layer will be absorbed and contribute to radiation pressure. The resulting force exerted on the matter is

$$F_{\text{rad}} = \frac{L}{c}$$

This has to be compared to the gravitational force

$$F_{\text{grav}} = \frac{G M_* m_{\text{layer}}}{r^2}$$

Using that $m_{\text{layer}} = 4\pi r^2 l \rho = 4\pi r^2 / \kappa_\nu$, we find that $F_{\text{rad}} = F_{\text{grav}}$ if the luminosity is equal to

$$L_{\text{Edd}} = \frac{4 \pi G M_* c}{\kappa_\nu}$$

which is called the **Eddington luminosity**. Stars with $L > L_{\text{Edd}}$ cannot exist, as they would blow themselves apart ($F_{\text{rad}} > F_{\text{grav}}$). The most massive stars known have luminosities that are very close to their Eddington luminosity.

Since the luminosities of AGN (supermassive black holes with accretion disks) are set by their accretion rate, the same argument implies an upper limit to the accretion rate of AGN, known as the **Eddington limit**.

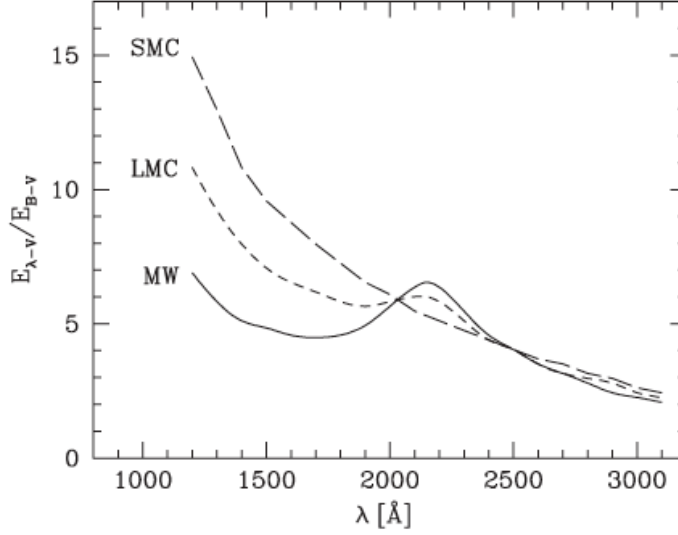


Figure 31: *Empirical extinction laws, defined in terms of the ratio of color excesses, for the Milky Way (MW), the Large Magellanic Cloud (LMC) and the Small Magellanic Cloud (SMC). Note the strong feature around 2100 Å in the MW extinction law, believed to be due to graphite dust grains.*

Extinction by Dust: Dust grains can scatter and absorb photons. Their ability to do so depends on (i) grain size, (ii) grain composition, and (iii) the presence of a magnetic field, which can cause grain alignment. Observationally, the extinction in the V-band is defined by

$$A_V \equiv -2.5 \log \left(\frac{f_V}{f_{V,0}} \right) = -2.5 \log \left(\frac{I_V}{I_{V,0}} \right)$$

where the subscript zero refers to the unextincted flux/intensity. Using that $I_V = I_{V,0} e^{-\tau_V}$ we have that

$$A_V = 1.086 \tau_V$$

More generally, $A_\lambda = 1.086 \tau_\lambda$; hence, an optical depth of unity roughly corresponds to an extinction of one magnitude.

Reddening: In addition to extinction, dust also causes reddening, due to the fact that dust extinction is more effective at shorter (bluer) wavelengths.

Color Excess: $E(B - V) \equiv A_B - A_V$, which can also be defined for any other wavebands.

Extinction law: conventionally, dust extinction is expressed in terms of an *empirical extinction law*:

$$k(\lambda) \equiv \frac{A_\lambda}{E(B - V)} \equiv R_V \frac{A_\lambda}{A_V}$$

where

$$R_V \equiv \frac{A_V}{E(B - V)}$$

is a quantity that is insensitive to the total amount of extinction; rather it expresses a property of the extinction law. Empirically, dust in the Milky Way seems to have $R_V \simeq 3.1$, while the dust in the Small Magellanic Cloud (SMC) is better characterized by $R_V \simeq 2.7$. Dust extinction law is not universal; rather, it is believed to depend on the local UV flux and the metallicity, among others (see Fig. 31).

Theoretical attempts to model the extinction curve have shown that dust comes in two varieties, graphites and silicates, while the grain-size distribution is well fit by $dN/da \propto a^{-3.5}$ and covers the range from $\sim 0.005\mu\text{m}$ to $\sim 0.25\mu\text{m}$. Note that for radiation with $\lambda > a_{\text{max}} \simeq 2500 \text{ \AA}$ dust mainly causes **Rayleigh scattering**.

CHAPTER 23

Radiative Transfer

Consider an incoming signal of **specific intensity** $I_{\nu,0}$ passing through a cloud (i.e., any gaseous region). As the radiation transits a small path length dr through the cloud, its specific intensity changes by $dI_{\nu} = dI_{\nu,\text{loss}} + dI_{\nu,\text{gain}}$. The loss-term describes the combined effect of scattering and absorption, which remove photons from the line-of-sight, while the gain-term describes all processes that add photons to the line-of-sight; these include all emission processes from the gas itself, as well as scattering of photons from any direction into the line-of-sight.

In what follows we ignore the contribution of scattering to $dI_{\nu,\text{gain}}$, as this term makes solving the equation of radiative transfer much more complicated. We will briefly comments on that below, but for now the only process that is assumed to contribute to $dI_{\nu,\text{gain}}$ are emission processes from the gas.

It is useful to define the following two coefficients:

- **Absorption coefficient**, $\alpha_{\nu} = n \sigma_{\nu} = \rho \kappa_{\nu}$, which has units $[\alpha_{\nu}] = \text{cm}^{-1}$.
- **Emission coefficient**, j_{ν} , defined as the energy emitted per unit time, per unit volume, per unit frequency, per unit solid angle (i.e., $dE = j_{\nu} dt dV d\nu d\Omega$, and thus $[j_{\nu}] = \text{erg s}^{-1} \text{cm}^{-3} \text{Hz}^{-1} \text{sr}^{-1}$).

In terms of these two coefficients, the **equation of radiative transfer** can be written in either of the following two forms:

$$\frac{dI_{\nu}}{dr} = -\alpha_{\nu} I_{\nu} + j_{\nu} \quad (\text{form I})$$

$$\frac{dI_{\nu}}{d\tau_{\nu}} = -I_{\nu} + S_{\nu} \quad (\text{form II})$$

Here $S_\nu \equiv j_\nu/\alpha_\nu$ is called the **source function**, and has units of specific intensity (i.e., $[S_\nu] = \text{erg s}^{-1} \text{cm}^{-2} \text{Hz}^{-1} \text{sr}^{-1}$). In order to derive form II from form I, recall that $d\tau_\nu = \alpha_\nu dr$ (see Chapter 25).

NOTE: we use the convention of τ_ν increasing *from* the source *towards* the observer. Some textbooks adopt the opposite convention, which results in some sign differences.

To get some insight, we now consider a number of different cases:

Case A No Cloud

In this case, there is no absorption ($\alpha_\nu = 0$) or emission ($j_\nu = 0$), other than the emission from the background source. Hence, we have that

$$\frac{dI_\nu}{dr} = 0 \quad \Rightarrow \quad I_\nu = I_{\nu,0}$$

which expresses that intensity is a conserved quantity in vacuum.

Case B Absorption Only

In this case, the cloud absorbs background radiation, but does not emit anything ($j_\nu = S_\nu = 0$). Hence,

$$\frac{dI_\nu}{d\tau_\nu} = -I_\nu$$

which is easily solved to yield

$$I_\nu = I_{\nu,0} e^{-\tau_\nu}$$

which is the expected result (see Chapter 25).

Case C Emission Only

If the cloud does not absorb ($\alpha_\nu = 0$) but does emit we have

$$\frac{dI_\nu}{dr} = j_\nu \quad \Rightarrow \quad I_\nu = I_{\nu,0} + \int_0^l j_\nu(r) dr$$

where l is the size of the cloud along the line-of-sight. This equation simply expresses that the *increase* of intensity is equal to the emission coefficient integrated along the line-of-sight.

Case D Cloud in Thermal Equilibrium w/o Background Source

Consider a cloud in TE, i.e., specified by a single temperature T (kinetic temperature is equal to radiation temperature). Since in a system in TE there can be no net transport of energy, we have that

$$\frac{dI_\nu}{dr} = -\alpha_\nu I_\nu + j_\nu = 0 \quad \Rightarrow \quad I_\nu = \frac{j_\nu}{\alpha_\nu} = S_\nu$$

Since the observer must see a black body of temperature T , we also have that $I_\nu = B_\nu(T)$ (i.e., the intensity is given by a Planck curve corresponding to the temperature of the cloud), and we thus have that

$$\begin{aligned} I_\nu &= S_\nu = B_\nu(T) \\ j_\nu &= \alpha_\nu B_\nu(T) \end{aligned}$$

The latter of these equivalent relations is sometimes called **Kirchoff's law**, and simply expresses that a black body needs to establish a balance between emission and absorption (i.e., $B_\nu(T) = j_\nu/\alpha_\nu$).

Case E Emission & Absorption (formal solution)

Consider the general case with both emission and absorption (but where we ignore the fact that scattering can scatter photons into my line of sight). Starting from form II of the equation of radiative transfer, multiplying both sides with e^{τ_ν} , we obtain that

$$\frac{d\tilde{I}_\nu}{d\tau_\nu} = -\tilde{S}_\nu$$

where $\tilde{I}_\nu \equiv I_\nu e^{\tau_\nu}$ and $\tilde{S}_\nu \equiv S_\nu e^{\tau_\nu}$. We can rewrite the above differential equation as

$$\int_{\tilde{I}_{\nu,0}}^{\tilde{I}_\nu} d\tilde{I}_\nu = \int_0^{\tau_\nu} \tilde{S}_\nu d\tau_\nu$$

Using that $\tilde{I}_{\nu,0} = I_{\nu,0} e^0 = I_{\nu,0}$ the solution to this simple integral equation is

$$\boxed{I_\nu = I_{\nu,0} e^{-\tau_\nu} + \int_0^{\tau_\nu} S_\nu(\tau'_\nu) e^{-(\tau_\nu - \tau'_\nu)} d\tau'_\nu}$$

where τ_ν is the total optical depth along the line of sight (i.e., through the cloud). The above is the formal solution, which, under the simplifying assumption that the **source function** is constant along the line of sight reduces to

$$\boxed{I_\nu = I_{\nu,0} e^{-\tau_\nu} + S_\nu (1 - e^{-\tau_\nu})}$$

The first term expresses **the attenuation of the background signal**, the second term expresses **the added signal due to the emission from the cloud**, while the third term describes **the cloud's self-absorption**.

Using the above formal solution to the equation of radiative transfer, we have the following two extremes:

$$\begin{aligned} \tau_\nu \gg 1 & \Rightarrow I_\nu = S_\nu \\ \tau_\nu \ll 1 & \Rightarrow I_\nu = I_{\nu,0} (1 - \tau_\nu) + S_\nu \tau_\nu \end{aligned}$$

where, for the latter case, we have used the Taylor series expansion for the exponential. In the high optical depth case, the observer just ‘sees’ the outer layers of the cloud, and therefore the observed intensity is simply the source function of the cloud (the observed signal contains no contribution from the background source). In the small optical depth limit, the contribution from the cloud is suppressed by a factor τ_ν , while that from the background source is attenuated by a factor $(1 - \tau_\nu)$. Using that $S_\nu = j_\nu/\alpha_\nu$ and $\tau_\nu = \alpha_\nu l$ (if the absorption coefficient is constant throughout the cloud), we see that $\tau_\nu S_\nu = j_\nu l$; in other words, the contribution from the cloud itself is simply its emission coefficient (assumed constant throughout the cloud) multiplied with the pathlength through the cloud.

To get some further insight into the source function and radiative transfer in general, consider form II of the radiative transfer equation. If $I_\nu > S_\nu$ then $dI_\nu/d\tau_\nu < 0$, so that the specific intensity *decreases* along the line of sight. If, on the other hand, $I_\nu < S_\nu$ then $dI_\nu/d\tau_\nu > 0$, indicating that the specific intensity *increases* along the line of sight. Hence, I_ν tends towards S_ν . If the optical depth of the cloud is sufficiently large then this ‘tendency’ will succeed, and $I_\nu = S_\nu$.

An important special case of the general solution derived above is if the cloud is in local thermal equilibrium (LTE). This is very often the case, since over the mean free path of the photons, every system will tend to be in LTE, unless it was recently

disturbed and has not yet been able to equilibrate. In the case of LTE, we have that, over a patch smaller than or equal to the mean free path of the photons, we have that $S_\nu \equiv j_\nu/\alpha_\nu = B_\nu(T)$, where T is the kinetic temperature (= radiation temperature) of the patch.

The solution to the **equation of radiative transfer** now is

$$I_\nu = I_{\nu,0} e^{-\tau_\nu} + B_\nu(T) [1 - e^{-\tau_\nu}]$$

Note that I_ν is **not** constant throughout the cloud, as was the case for a cloud in TE. In the case of LTE, however, there can be a non-zero gradient dI_ν/dr .

Before we interpret this result in detail, it is important to distinguish

Blackbody Radiation: $I_\nu = B_\nu(T)$

Thermal Radiation: $S_\nu = B_\nu(T)$

NOTE: thermal radiation is radiation emitted by matter in thermal equilibrium.

Keeping this difference in mind, we now look at the solution to our equation of radiative transfer for a cloud in LTE at its two extremes:

$$\begin{aligned} \tau_\nu \gg 1 & \Rightarrow I_\nu = B_\nu(T) \\ \tau_\nu \ll 1 & \Rightarrow I_\nu = I_{\nu,0} (1 - \tau_\nu) + B_\nu(T) \tau_\nu \end{aligned}$$

The former expresses that *an optically thick cloud in LTE emits black body radiation*. This is characterized by the fact that (i) if there is a background source, you can't see it, (ii) you can look into the source only for about one mean free path of the photons (which is much smaller than the size of the source), and (iii) the only information available to an observer is the temperature of the cloud (the observed intensity is a Planck curve of temperature T).

A good example of gas clouds in LTE are stars!

In the optically thin limit, the observed intensity depends on the background source (if present), and depends on *both* the temperature (sets source function) and density (sets optical depth) of the cloud (recall that $\tau_\nu \propto \kappa_\nu \rho l$).

In the case without background source we have that

$$I_\nu = \begin{cases} B_\nu(T) & \text{if } \tau_\nu \gg 1 \\ \tau_\nu B_\nu(T) & \text{if } \tau_\nu \ll 1 \end{cases}$$

Note that this is different from **case D**, in which we considered a cloud in TE without background source. In that case we obtained that $I_\nu = B_\nu(T)$ *independent of* τ_ν . In the case of LTE, however, there are radial gradients, which are responsible for diminishing the intensity by the optical depth in the case where $\tau_\nu \ll 1$. This may seem somewhat ‘counter-intuitive’, as it indicates that a cloud of larger optical depth is more intense!!! To understand this, consider the limit $\tau_\nu \rightarrow 0$. In this case all photons pass through the cloud with zero probability to be absorbed/scattered. In this situation, there is simply no way to establish an equilibrium between emission and absorption required for the establishment of a black body; or, put differently, if there is no absorption, there is no emission either (after all, we are in LTE), and thus, $I_\nu = 0$.

Based on the above, we have that, in the case of a cloud in LTE without background source, $I_\nu \leq B_\nu(T)$, where T is the temperature of the cloud. If we express the intensity in terms of the **brightness temperature** we have that $T_{B,\nu} \leq T$. Hence, for a cloud in LTE without background source the observed brightness temperature is a *lower limit* on the kinetic temperature of the cloud.

What about scattering? In the most general case, any element in the cloud receives radiation coming from all 4π steradian, and a certain fraction of that radiation will be scattered into the line-of-sight of an observer.

In general, the scattering can (will) be **non-isotropic** (e.g., Thomson scattering) and **incoherent** (e.g., Compton scattering or resonant scattering), and the final equation of radiative transfer can only be solved numerically.

In the simplified case of **isotropic, coherent** scattering the corresponding emission coefficient can be found by simply equating the power absorbed per unit volume to that emitted (for each frequency);

$$j_{\nu,\text{scat}} = \alpha_{\nu,\text{scat}} J_\nu$$

where $\alpha_{\nu,\text{scat}}$ is the **absorption coefficient** of the scattering processes, while

$$J_\nu = \frac{1}{4\pi} \int I_\nu d\Omega$$

is the **mean intensity**, averaged over all 4π steradian.

The **source function** due to scattering is then simply

$$S_\nu \equiv \frac{j_{\nu,\text{scat}}}{\alpha_{\nu,\text{scat}}} = J_\nu = \frac{1}{4\pi} \int I_\nu d\Omega$$

Hence, the source function due to **isotropic, coherent scattering** is simply the mean intensity.

The radiative transfer equation for pure scattering (no background source, and no emission) is

$$\frac{dI_\nu}{dr} = -\alpha_{\nu,\text{scat}} (I_\nu - J_\nu)$$

Even this oversimplified case of pure isotropic, coherent scattering is not easily solved. Since J_ν involves an integration (over all 4π steradian), the above equation is an integro-differential equation, which are extremely difficult to solve in general; one typically has to resort to numerical methods (see Rybicki & Lightmann 1979 for more details). **NOTE:** although the scattering may be isotropic, the incoming radiation is typically not.

Observability of Emission & Absorption Lines: Consider a cloud in front of some background source. Assume the cloud is in LTE at temperature T . Assume that α_ν is only non-zero at a specific frequency, ν_1 , corresponding to some electron transition. Given that resonant scattering is typically orders of magnitude more efficient than other scattering mechanisms, this is a reasonable approximation. The intensity observed is

$$I_\nu = I_{\nu,0} e^{-\tau_\nu} + B_\nu(T) [1 - e^{-\tau_\nu}]$$

At all frequencies other than ν_1 we have $\tau_\nu = 0$, and thus $I_\nu = I_{\nu,0}$. Now assume that the observer sees an **absorption line** at $\nu = \nu_1$. This implies that

$$I_{\nu_1} = I_{\nu_1,0} e^{-\tau_{\nu_1}} + B_{\nu_1}(T) [1 - e^{-\tau_{\nu_1}}] < I_{\nu_1,0}$$

while in the case of an **emission line**

$$I_{\nu_1} = I_{\nu_1,0} e^{-\tau_{\nu_1}} + B_{\nu_1}(T) [1 - e^{-\tau_{\nu_1}}] > I_{\nu_1,0}$$

Rearranging, we then have that

Absorption Line:	$B_{\nu}(T) < I_{\nu,0}$	$T < T_{B,\nu}$
Emission Line:	$B_{\nu}(T) > I_{\nu,0}$	$T > T_{B,\nu}$

where T is the (kinetic) temperature of the cloud, and $T_{B,\nu}$ is the **brightness temperature** of the background source, at the frequency of the line. Hence, if the cloud is colder (hotter) than the source, an absorption (emission) line will arise. In the case of no background source, we effectively have that $T_{B,\nu} = 0$, and the cloud will thus reveal an emission line. In the case where $T = T_{B,\nu}$ no line will be visible, independent of the optical depth of the cloud!

CHAPTER 24

Continuum Emission Mechanisms

Continuum radiation is any radiation that forms a continuous spectrum and is not restricted to a narrow frequency range. In what follows we briefly describe five continuum emission mechanisms:

- **Thermal (Black Body) Radiation**
- **Bremsstrahlung (free-free emission)**
- **Recombination (free-bound emission)**
- **Two-Photon emission**
- **Synchrotron emission**

In general, the way to proceed is to ‘derive’ the emission coefficient, j_ν , the absorption coefficient, α_ν , and then use the equation of **radiative transfer** to compute the specific intensity, I_ν , (i.e., the ‘spectrum’), for a cloud of gas emitting continuum radiation using any one of those mechanisms.

First some general remarks: when talking about continuum processes it is important to distinguish **thermal emission**, in which the radiation is generated by the thermal motion of charged particles and in which the intensity therefore depends (at least) on temperature, i.e., $I_\nu = I_\nu(T, ..)$, from **non-thermal emission**, which is everything else.

Examples of **thermal continuum emission** are *black body radiation* and *(thermal) bremsstrahlung*, while *synchrotron radiation* is an example of *non-thermal emission*. Another non-thermal continuum mechanism is *inverse compton radiation*. However, since this is basically an incoherent photon-scattering mechanism, rather than a photon-production mechanism, we will not discuss IC scattering any further here.

Characteristics of Thermal Continuum Emission:

- **Low Brightness Temperatures:** Since one rarely encounters gases with kinetic temperatures $T > 10^7 - 10^8 \text{K}$, and since $T_{\text{B}} \leq T$ (see Chapter 28), if the brightness temperature of the radiation exceeds $\sim 10^8 \text{K}$ it is most likely non-thermal in origin (or has experienced IC scattering).

- **No Polarization:** Since there is no particular directionality to the thermal motion of particles, thermal emission is essentially unpolarized. In other words, if emission is found to be polarized, it is either non-thermal, or the signal became polarized after it was emitted (i.e., via Thomson scattering).

Thermal Radiation & Black Body Radiation: Thermal radiation is the continuum emission arising from particles colliding, which causes acceleration of charges (atoms typically have electric or magnetic dipole moments, and colliding those results in the emission of photons). This thermal radiation tries to establish **thermal equilibrium** with the matter that produces it via photon-matter interactions. If thermal equilibrium is established (locally), then the **source function** $S_{\nu} \equiv j_{\nu}/\alpha_{\nu} = B_{\nu}(T)$ (**Kirchoff's law**).

As we have seen in the previous Chapter;

$$I_{\nu} = \begin{cases} B_{\nu}(T) & \text{if } \tau_{\nu} \gg 1 \\ \tau_{\nu} B_{\nu}(T) & \text{if } \tau_{\nu} \ll 1 \end{cases}$$

where $\tau_{\nu} = \alpha_{\nu} l$ is the **optical depth** through the cloud, which has a dimension l along the line-of-sight.

Free-free emission (Bremsstrahlung): Bremsstrahlung (German for ‘braking radiation’) arises when a charged particle (i.e., an electron) is accelerated through the Coulomb interaction with another charged particle (i.e., an ion of charge Ze). Effectively what happens is that the two charges make up an electric dipole which, due to the motion of the charges, is time variable. A variable dipole is basically an antenna, and emits electromagnetic waves. The energy in these EM waves (photons) emitted is lost to the electron, which therefore loses (kinetic) energy (the electron is ‘braking’).

It is fairly straightforward to compute the amount of energy radiated by a single

electron moving with velocity v when experiencing a Coulomb interaction with a charge Ze over an impact parameter b (see Rybicki & Lightmann 1979 for a detailed derivation).

The next step is to integrate over all possible impact parameters. This are all impact parameters $b > b_{\min}$, where from a classical perspective $b_{\min}$ is set by the requirement that the kinetic energy of the electron, $E_k = \frac{1}{2} m_e v^2$, is larger than the binding energy, $E_b = Ze^2/b$ (otherwise we are in the regime of **recombination**; see below). However, there are some quantum mechanical corrections one needs to make to this $b_{\min}$ which arise from Heisenberg's Uncertainty Principle ($\Delta x \Delta p \geq \hbar/2$). This correction factor is called the **free-free Gaunt factor**, $g_{\text{ff}}(\nu, T_e)$, which is close to unity, and has only a weak frequency dependence. The final step in obtaining the emission coefficient is the integration over the **Maxwellian velocity distribution** of the electrons, characterized by T_e . The result (in $\text{erg s}^{-1} \text{cm}^{-3} \text{Hz}^{-1} \text{sr}^{-1}$) is:

$$j_\nu = 5.44 \times 10^{-39} \left(\frac{Z^2}{T_e^{1/2}} \right) n_e n_i g_{\text{ff}}(\nu, T_e) e^{-h\nu/k_B T_e}$$

In the case of a pure (ionized) hydrogen gas, $Z = 1$ and $n_i = n_e$. Upon inspection, it is clear that free-free emission has a flat spectrum $j_\nu \propto \nu^\alpha$ with $\alpha \sim 0$ (controlled by the weak frequency dependence of the Gaunt factor) with an exponential cut-off for $h\nu > k_B T_e$ (the maximum photon energy is set by the temperature of the electrons). This reveals that a measurement of the exponential cut-off is a direct measure of the electron temperature.

The above emission coefficient tells us the emissive behavior of a pocket of (ionized) gas without allowance for the internal absorption. Accounting for the latter requires radiative transfer. Since Bremsstrahlung arises from collisions, we may use the LTE approximation. Hence, **Kirchoff's law** tells us that $\alpha_\nu = j_\nu/B_\nu(T)$, which allows us to compute the absorption coefficient, and thus the optical depth $\tau_\nu = \alpha_\nu l$. Substitution of $B_\nu(T)$, with $T = T_e$, yields

$$\tau_\nu \simeq 3.7 \times 10^8 Z^2 T_e^{-1/2} \nu^{-3} [1 - e^{-h\nu/k_B T_e}] g_{\text{ff}}(\nu, T_e) \mathcal{E}$$

where

$$\mathcal{E} \equiv \int n_e^2 dl \simeq n_e^2 l$$

is called the **emission measure**, and we have assumed that $n_e = n_i$. Upon inspection, one notices that $\tau_\nu \propto \nu^{-2}$ (for $h\nu \ll k_B T_e$), indicating that the opacity of the cloud increases with decreasing frequency. This opacity arises from **free-free absorption**, which is simply the inverse process of free-free emission; a photon is absorbed by an electron that is experiencing a Coulomb interaction.

If we now substitute our results in the **equation of radiative transfer** (without background source),

$$I_\nu = B_\nu(T) [1 - e^{-\tau_\nu}]$$

then we obtain that

$$I_\nu = \begin{cases} B_\nu(T_e) & \text{if } \tau_\nu \gg 1 \\ \tau_\nu B_\nu(T) = j_\nu l & \text{if } \tau_\nu \ll 1 \end{cases}$$

Fig. 32 shows an illustration of a typical free-free emission spectrum: at low frequency the gas is optically thick, and one probes the Rayleigh-Jeans part of the Planck curve corresponding to the electron temperature ($I_\nu \propto \nu^2 T_e$). At intermediate frequencies, where the cloud is optically thin, the spectrum is flat ($I_\nu \propto \mathcal{E} T_e^{-1/2}$), and at the high-frequency end there is an exponential cut-off ($I_\nu \propto \exp[-h\nu/k_B T_e]$).

Free-Bound emission (Recombination): this involves the capture of a free electron by a nucleus into a quantized bound state. Hence, this requires the medium to be ionized, similar to free-free emission, and in general both will occur (complicating the picture). Free-bound emission is basically the same as free-free emission (they have the same emission coefficient, j_ν), except that they involve different integration ranges for the impact parameter b , and therefore different Gaunt factors; the **free-bound Gaunt factor** $g_{fb}(\nu, T_e)$ has a different temperature dependence than $g_{ff}(\nu, T_e)$, and also has more ‘structure’ in its frequency dependence; in the limit where the bound state has a large quantum number (i.e., the electron is weakly bound), we have that $g_{fb} \sim g_{ff}$. However, for more bound states the frequency dependence of g_{fb} reveals sharp ‘edges’ associated with the discrete bound states.

When $k_B T_e \gg h\nu$ recombination is negligible (electrons are moving too fast to become bound), and the emission is dominated by the free-free process (i.e., $g_{fb}(\nu, T_e) \rightarrow 0$ for $h\nu \ll k_B T_e$). At lower electron temperatures (or, equivalently, higher photon frequencies), recombination becomes more and more important, and often will dominate over bremsstrahlung, $g_{fb}(\nu, T_e) > g_{ff}(\nu, T_e)$, (see Fig. 33).

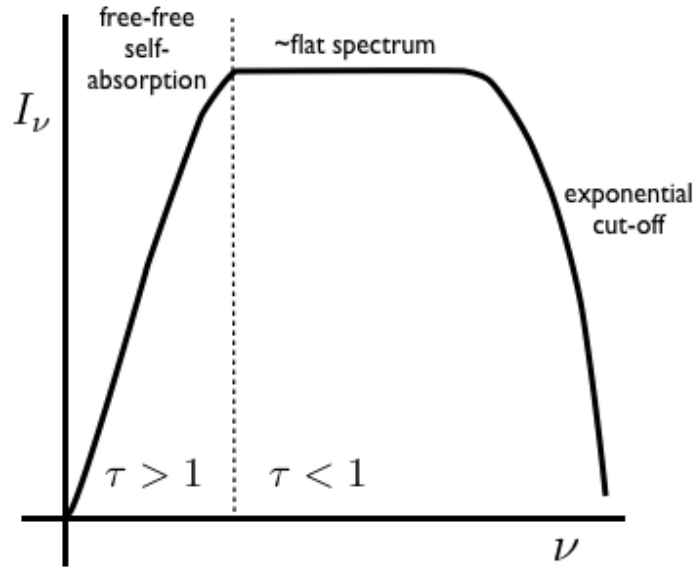


Figure 32: *Specific intensity of free-free emission (Bremsstrahlung), including the effect of free-free self absorption at low frequencies, where the optical depth exceeds unity. At low frequencies, one probes the Rayleigh-Jeans part of the Planck curve corresponding to the electron temperature. At intermediate frequencies, where the cloud is optically thin, the spectrum is flat, followed by an exponential cut-off at the high-frequency end.*

Two-Photon Emission: two photon emission occurs between *bound* states in an atom, but it produces *continuum* emission rather than line emission.

Two photon emission occurs when an electron finds itself in a quantum level for which *any* downward transition would violate quantum mechanical selection rules. Each transition is therefore highly forbidden. However, there is a non-zero chance that the electron decays under the emission of two, rather than one, photons. Energy conservation guarantees that $\nu_1 + \nu_2 = \nu_{\text{tr}} = \Delta E_{\text{tr}}/h$, where ΔE_{tr} is the energy difference associated with the transition. The *most probable* configuration is the one in which $\nu_1 = \nu_2 = \nu_{\text{tr}}/2$, but all configurations that satisfy the above energy conservation are possible; they become less likely the larger $|\nu_1 - \nu_{\text{tr}}/2|$, resulting in a ‘continuum’ emission that appears as an extremely broad ‘emission line’. In fact, whereas the *number* of photons with $0 < \nu < \nu_{\text{tr}}/2$ is equal to that with $\nu_{\text{tr}}/2 < \nu < \nu_{\text{tr}}$,

the latter have more *energy* (i.e., $E_\gamma = h\nu$). Consequently, the spectral energy distribution, L_ν (erg s⁻¹ Hz⁻¹) is skewed towards higher frequency.

For two photon emission to occur, we require that spontaneous emission happens before collisional de-excitation has a chance. Consequently, two-photon emission occurs in low density ionized gas. The strength of the two photon emission depends on the number of particles in the excited states. This in turn depends on the recombination rate; although two-photon emission is quantum-mechanical in nature, it can still be thought of as ‘thermal emission’, and the density dependence is the same as for free-free and free-bound emission (i.e., $j_\nu \propto n_e^2$).

An important example of two-photon emission is associated with the Ly α recombination line, which results from a de-excitation of an electron from the $n = 2$ to $n = 1$ energy level in a Hydrogen atom. As it turns out, the $n = 2$ quantum level consists of both $2s$ and $2p$ states. The transition $2p \rightarrow 1s$ is a permitted transition with $A_{2p \rightarrow 1s} = 6.27 \times 10^8 \text{s}^{-1}$. However, the $2s \rightarrow 1s$ transition is highly forbidden, and has a two-photon-emission rate coefficient of $A_{2s \rightarrow 1s} = 8.2 \text{s}^{-1}$. Although this is orders of magnitude lower than for the $2p \rightarrow 1s$ transition, the two photon emission can still be important in low-density nebulae ($n \lesssim 10^4 \text{cm}^{-3}$).

Synchrotron & Cyclotron Emission: A free electron moving in a magnetic field experiences a **Lorentz force**:

$$\vec{F}_e = e \left(\frac{\vec{v}}{c} \times \vec{B} \right) = \frac{ev}{c} B \sin \phi = \frac{ev}{c} B_\perp = \frac{ev_\perp}{c} B$$

where ϕ is the *pitch angle* between $\vec{v}$ and $\vec{B}$. If $\phi = 0$ the particle moves along the magnetic field, and the Lorentz force is zero. If $\phi = 90^\circ$ the particle will move in a circle around the magnetic field line, while for $0^\circ < \phi < 90^\circ$ the electron will spiral (‘cork-screw’) around the magnetic field line. In the latter two cases, the electron is being accelerated, which causes the emission of photons. Note that this applies to both electrons and ions. However, since the cyclotron (synchrotron) emission from ions is negligible compared to that from electrons, we will focus on the latter.

If the particle is *non-relativistic*, then the emission is called **cyclotron emission**. If, on the other hand, the particles are *relativistic*, the emission is called **synchrotron emission**. We will first focus on the former.

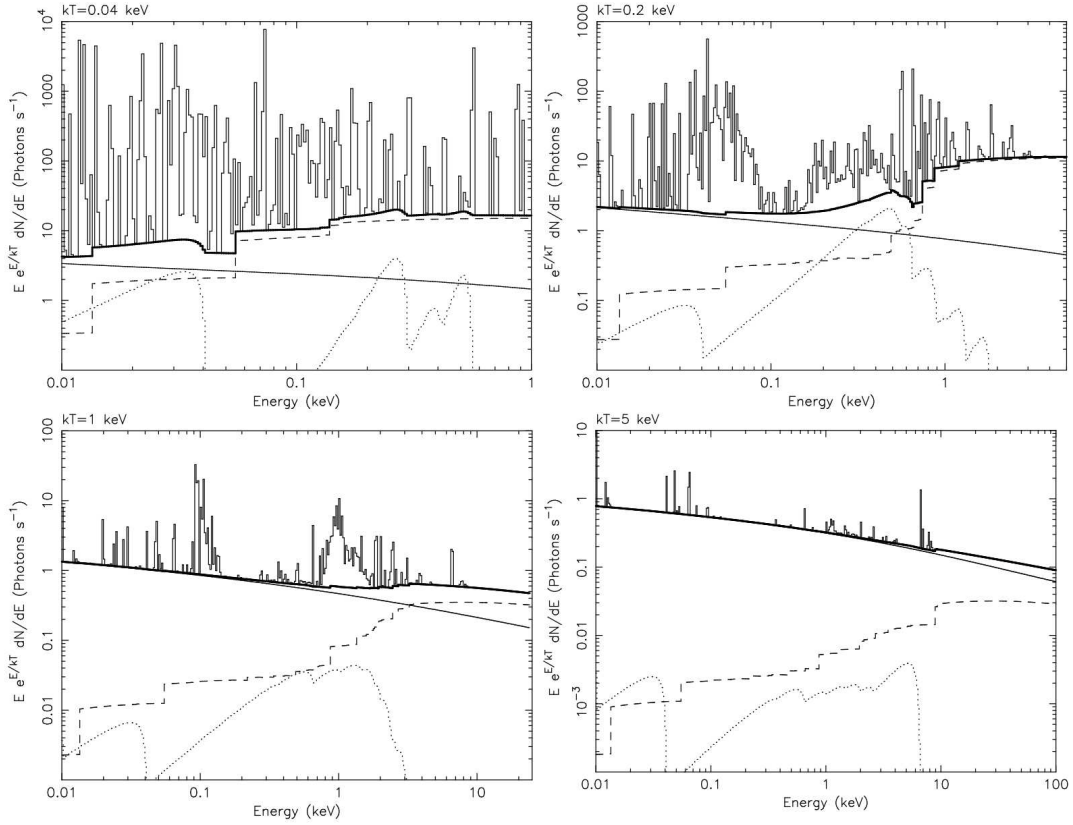


Figure 33: *Emission spectra of plasmas with solar abundances. The histogram indicates the total spectrum, including line radiation. The spectrum has been binned in order to highlight the relative importance of line radiation. The thick solid line is the total continuum emission, the thin solid line the contribution due to Bremsstrahlung, the dashed line free-bound emission and the dotted line two-photon emission. Note how recombination becomes less and less important when the gas gets hotter. [From Kaastra et al. 2008, Space Science Reviews, **134**, 155]*

Cyclotron emission: the gyrating electron emits dipolar emission that (i) has the frequency of **gyration**, and (ii) is highly polarized. Depending on the viewing angle the observer can see circular polarization (if line-of-sight is aligned with $\vec{B}$), linear polarization, if line of sight is perpendicular to $\vec{B}$, or elliptical polarization (for any other orientation).

The gyration frequency can be obtained by equating the Lorentz force with the centripetal force:

$$F_e = \frac{e v_{\perp}}{c} B = \frac{m_e v_{\perp}^2}{r_0}$$

where $v_{\perp} = v \sin \phi$, which results in

$$r_0 = \frac{m_e v_{\perp} c}{e B}$$

where $B = |\vec{B}|$. This is called the **gyration radius** (or gyro-radius). The period of gyration is $T = 2\pi r_0 / v_{\perp}$, which implies a **gyration frequency** (i.e., the frequency of the emitted photons) of

$$\nu_0 = \frac{1}{T} = \frac{e B}{2\pi m_e c}$$

Note that this frequency is *independent of the velocity of the electron*! It only depends on the magnetic field strength B ;

$$\boxed{\boxed{\frac{\nu_0}{\text{MHz}} = 2.8 \frac{|\vec{B}|}{\text{Gauss}}}}$$

We thus see that **cyclotron emission** really is line emission, rather than continuum emission. The nature of this line emission is very different though, from ‘normal’ spectral lines which result from quantum transitions within atoms or molecules. Note, though, that if the ‘source’ has a varying magnetic field, then the variance in B will result in a ‘broadening’ of the line, which, if sufficiently large, may appear as ‘continuum emission’.

In principle, observing cyclotron emission immediately yields the magnetic field strength. However, unless B is extremely large, the frequency of the cyclotron emission is extremely low; typical magnetic field strengths in the IGM are of the order of

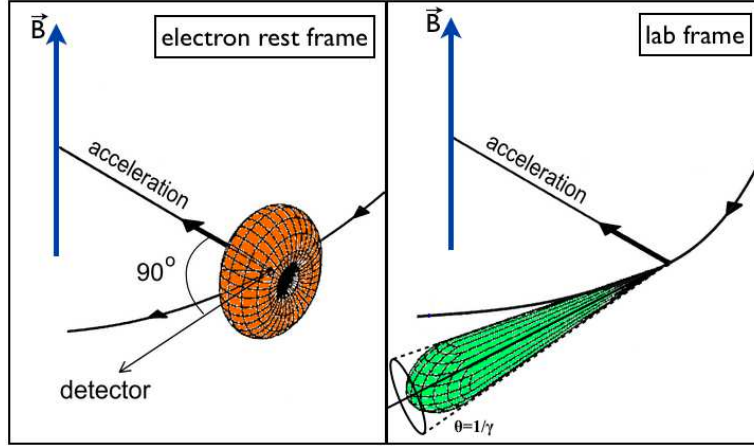


Figure 34: *Illustration of how the Lorentz transformation from the electron rest frame to the lab frame introduce relativistic beaming with an opening angle $\theta = 1/\gamma$. Note that in the electron rest frame, the synchrotron emission is dipole emission.*

several μG , which implies cyclotron frequencies in the few Hz regime. The problem is that such low frequency radiation will not be able to travel through an astrophysical plasma, because the frequency is lower than the **plasma frequency**, which is the natural frequency of a plasma (see Chapter 20). In addition, the Earth's ionosphere blocks radiation with a frequency $\nu \lesssim 10\text{MHz}$, so that we can only observe cyclotron emission from the Earth's surface if it originates from objects with $B \gtrsim 3.5\text{G}$. For this reason, cyclotron emission is rarely observed, with the exception of the Sun, some of the planets in our Solar System, and an occasional pulsar.

Synchrotron Emission: this is the same as cyclotron emission, but in the limit in which the electrons are relativistic. As we demonstrate below, this has two important effects: it makes the gyration frequency dependent on the energy (velocity) of the electron, and it causes strong beaming of the electron's dipole emission.

In the relativistic regime, the electron energy becomes $E_e = \gamma m_e c^2$, where $\gamma = (1 - v^2/c^2)^{-1/2}$ is the **Lorentz factor**. This boosts the gyration radius by a factor

γ , and reduces the gyration frequency by $1/\gamma$:

$$\begin{aligned} r_0 &= \frac{\gamma m_e v_\perp c}{e B} \simeq \frac{\gamma m_e c^2}{e B} \\ \nu_0 &= \frac{e B}{2\pi \gamma m_e c} \end{aligned}$$

Note that now the gyration frequency *does* depend on the velocity (energy) of the (relativistic) electrons, which in principle implies that because the electrons will have a distribution in energies, the synchrotron emission is going to be continuum emission. However, you can also see that the gyration frequency is even **lower** than in the case of cyclotron emission, by a factor $1/\gamma$. For the record, Lorentz factors of up to $\sim 10^{11}$ have been measured, indicating that γ can be extremely large! Hence, if the photon emission were to be at the gyration frequency, we would never be able to see it, because of the plasme-frequency-shielding.

However, the gyration frequency is not the only frequency in this problem. Because of the relativistic motion, the dipole emission from the electron, as seen from the observer's frame, is highly beamed (see Fig. 34), with an opening angle $\sim 1/\gamma$ (which can thus be tiny). Consequently, the observer does not have a continuous view of the electron, but only sees EM radiation when the beam sweeps over the line-of-sight. The width of these 'pulses' are a factor $1/\gamma^3$ shorter than the gyration period. The corresponding frequency, called the **critical frequency**, is given by

$$\nu_{\text{crit}} = \frac{3e}{4\pi m_e c} \gamma^2 B_\perp$$

which translates into

$$\boxed{\frac{\nu_{\text{crit}}}{\text{MHz}} = 4.2 \gamma^2 \frac{B_\perp}{\text{Gauss}}}$$

So although the gyration frequency will be small, the critical frequency can be extremely large. This critical frequency corresponds to the shortest time period (the pulse duration), and therefore represents the largest frequency, above which the emission is negligible. The longest time period, which is related to the gyration period, determines the **fundamental frequency**

$$\boxed{\frac{\nu_f}{\text{MHz}} = \frac{2.8}{\gamma \sin^2 \phi} \frac{|\vec{B}|}{\text{Gauss}}}$$

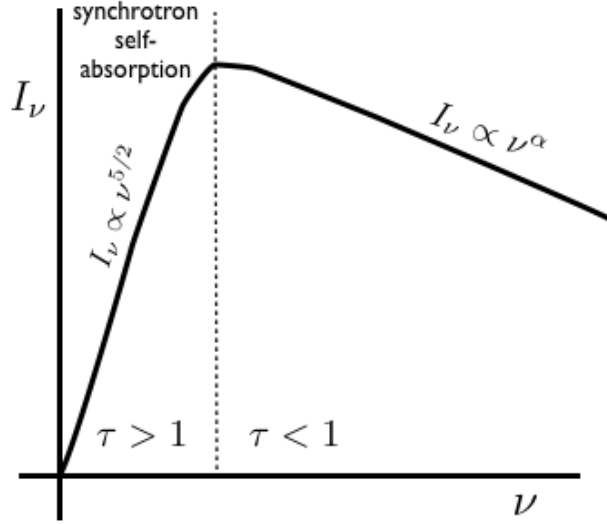


Figure 35: *Specific intensity of synchrotron emission, including the effect of synchrotron self absorption at low frequencies, where the optical depth exceeds unity.*

The emission spectrum due to synchrotron radiation will contain this fundamental frequency plus all its harmonics up to ν_{crit} . Since these harmonics are extremely closely spaced (after all, the gyration frequency is extremely small), the synchrotron spectrum *for one value of γ* looks essentially continuum. When taking the γ -distribution into account (which is related to the energy distribution of the relativistic electrons), the distribution becomes truly continuum, and the critical and fundamental frequencies no longer can be discerned (because they depend on γ).

After integrating over the energy distribution of the relativistic electrons, which typically has a power-law distribution $N(E) \propto E^{-\Gamma}$ one obtains the following **emission and absorption coefficients**:

$$\begin{aligned} j_\nu &\propto B_\perp^{(\Gamma+1)/2} \nu^{-(\Gamma-1)/2} \\ \alpha_\nu &\propto B_\perp^{(\Gamma+2)/2} \nu^{-(\Gamma+4)/2} \end{aligned}$$

Note that α_ν describes **synchrotron self-absorption**. The resulting **source func-**

tion and optical depth are

$$S_\nu = \frac{j_\nu}{\alpha_\nu} \propto B_\perp^{-1/2} \nu^{5/2}$$

$$\tau_\nu = \alpha_\nu l \propto B_\perp^{(\Gamma+2)/2} \nu^{-(\Gamma+4)/2} l$$

Using that typically $\Gamma > 0$, we have that $\tau_\nu \propto \nu^a$ with $a < 0$; synchrotron self-absorption becomes more important at lower frequencies.

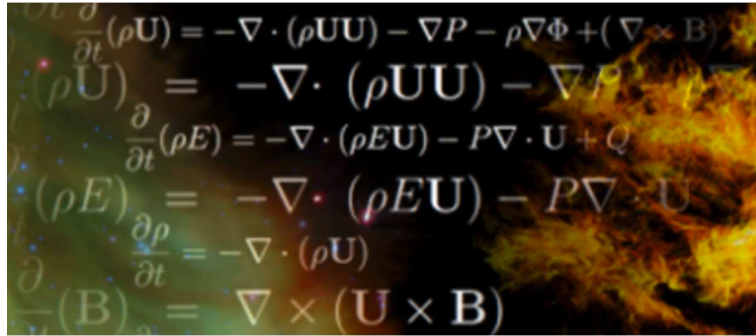
Application of the equation of radiative transfer, $I_\nu = S_\nu (1 - e^{-\tau_\nu})$, yields

$$I_\nu = \begin{cases} S_\nu \propto \nu^{5/2} & \text{if } \tau_\nu \gg 1 \\ j_\nu l \propto \nu^\alpha & \text{if } \tau_\nu \ll 1 \end{cases}$$

where $\alpha \equiv -\frac{\Gamma-1}{2}$. Fig. 35 shown an illustration of a typical synchrotron spectrum: at low frequencies, where $\tau_\nu \gg 1$, we have that $I_\nu \propto \nu^{5/2}$, which transits to $I_\nu \propto \nu^{-(\Gamma-1)/2}$ once the emitting medium becomes optically thin for synchrotron self-absorption. Note that there is no cut-off related to the critical frequency, since $\nu_{\text{crit}} = \nu_{\text{crit}}(E)$.

Supplemental Material

Appendices



Appendices A-E present background material on calculus relevant for this course. The other Appendices present supplemental material that is NOT considered part of this course's curriculum. They are included to provide background information for those readers that want to know a bit more.

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Appendix A

Vector Calculus

Vector: $\vec{A} = (a_1, a_2, a_3) = a_1\hat{i} + a_2\hat{j} + a_3\hat{k}$

Amplitude of vector: $|\vec{A}| = \sqrt{a_1^2 + a_2^2 + a_3^2}$

Unit vector: $|\vec{A}| = 1$

Basis: In the above example, the unit vectors $\hat{i}$, $\hat{j}$ and $\hat{k}$ form a vector basis.
Any 3 vectors $\vec{A}$, $\vec{B}$ and $\vec{C}$ can form a vector basis
as long as $\det(\vec{A}, \vec{B}, \vec{C}) \neq 0$.

Determinant: $\det(\vec{A}, \vec{B}) = \begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix} = a_1b_2 - a_2b_1$

$$\det(\vec{A}, \vec{B}, \vec{C}) = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} = a_1 \begin{vmatrix} b_2 & b_3 \\ c_2 & c_3 \end{vmatrix} + a_2 \begin{vmatrix} b_3 & b_1 \\ c_3 & c_1 \end{vmatrix} + a_3 \begin{vmatrix} b_1 & b_2 \\ c_1 & c_2 \end{vmatrix}$$

Geometrically: $\det(\vec{A}, \vec{B}) = \pm \text{area of parallelogram}$
 $\det(\vec{A}, \vec{B}, \vec{C}) = \pm \text{volume of parallelepiped}$

Multiplication by scalar: $\alpha \vec{A} = (\alpha a_1, \alpha a_2, \alpha a_3)$
 $|\alpha \vec{A}| = |\alpha| |\vec{A}|$

Summation of vectors: $\vec{A} + \vec{B} = \vec{B} + \vec{A} = (a_1 + b_1, a_2 + b_2, a_3 + b_3)$

Einstein Summation Convention: $a_i b_i = \sum_i a_i b_i = a_1 b_1 + a_2 b_2 + a_3 b_3 = \vec{a} \cdot \vec{b}$
 $\partial A_i / \partial x_i = \partial A_1 / \partial x_1 + \partial A_2 / \partial x_2 + \partial A_3 / \partial x_3 = \nabla \cdot \vec{A}$
 $A_{ii} = A_{11} + A_{22} + A_{33} = \text{Tr } \vec{A}$ (trace of $\vec{A}$)

Dot product (aka scalar product): $\vec{A} \cdot \vec{B} = a_i b_i = |\vec{A}| |\vec{B}| \cos \theta$
 $\vec{A} \cdot \vec{B} = \vec{B} \cdot \vec{A}$

Useful for:

- computing angle between two vectors: $\cos \theta = \vec{A} \cdot \vec{B} / (|\vec{A}| |\vec{B}|)$
- check orthogonality: two vectors are orthogonal if $\vec{A} \cdot \vec{B} = 0$
- compute projection of $\vec{B}$ in direction of $\vec{A}$, which is given by $\vec{A} \cdot \vec{B} / |\vec{A}|$

Cross Product (aka vector product): $\vec{A} \times \vec{B} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix} = \varepsilon_{ijk} a_i b_j \hat{e}_k$
 $|\vec{A} \times \vec{B}| = |\vec{A}| |\vec{B}| \sin \theta = \det(\vec{A}, \vec{B})$

NOTE: ε_{ijk} is called the **Levi-Civita tensor**, which is described in Appendix E,.

In addition to the dot product and cross product, there is a third vector product that one occasionally encounters in dynamics;

Tensor product: $\vec{A} \otimes \vec{B} = \overline{\overline{\mathbf{AB}}}$ $(\overline{\overline{\mathbf{AB}}})_{ij} = a_i b_j$
 $\vec{A} \otimes \vec{B} \neq \vec{B} \otimes \vec{A}$

The tensor product $\overline{\overline{\mathbf{AB}}}$ is a tensor of rank two and is called a **dyad**. The sum of two or more **dyads** is called a **dyadic**. For example, let $\vec{A}$, $\vec{B}$, $\vec{C}$ and $\vec{D}$ be four vectors, from which we can form the dyads $\overline{\overline{\mathbf{AB}}}$ and $\overline{\overline{\mathbf{CD}}}$. Their sum $\overline{\overline{\mathbf{AB}}} + \overline{\overline{\mathbf{CD}}}$ is then a **dyadic**. Note that in general a **dyadic** is not a **dyad** because it cannot be written as a vector multiplied with a vector. Hence, dyadics differ from vectors in that the sum of two vectors is a vector whereas the sum of two dyads is not necessarily a dyad.

$\vec{A} \cdot \vec{B} = \vec{B} \cdot \vec{A}$	$\vec{A} \times \vec{B} = -\vec{B} \times \vec{A}$
$(\alpha \vec{A}) \cdot \vec{B} = \alpha(\vec{A} \cdot \vec{B}) = \vec{A} \cdot (\alpha \vec{B})$	$(\alpha \vec{A}) \times \vec{B} = \alpha(\vec{A} \times \vec{B}) = \vec{A} \times (\alpha \vec{B})$
$\vec{A} \cdot (\vec{B} + \vec{C}) = \vec{A} \cdot \vec{B} + \vec{A} \cdot \vec{C}$	$\vec{A} \times (\vec{B} + \vec{C}) = \vec{A} \times \vec{B} + \vec{A} \times \vec{C}$
$\vec{A} \cdot \vec{B} = 0 \rightarrow \vec{A} \perp \vec{B}$	$\vec{A} \times \vec{B} = 0 \rightarrow \vec{A} \parallel \vec{B}$
$\vec{A} \cdot \vec{A} = \vec{A} ^2$	$\vec{A} \times \vec{A} = 0$

Triple Scalar Product: $\vec{A} \cdot (\vec{B} \times \vec{C}) = \det(\vec{A}, \vec{B}, \vec{C}) = \varepsilon_{ijk} a_i b_j c_k$
 $\vec{A} \cdot (\vec{B} \times \vec{C}) = 0 \rightarrow \vec{A}, \vec{B}, \vec{C}$ are coplanar
 $\vec{A} \cdot (\vec{B} \times \vec{C}) = \vec{B} \cdot (\vec{C} \times \vec{A}) = \vec{C} \cdot (\vec{A} \times \vec{B})$

Triple Vector Product: $\vec{A} \times (\vec{B} \times \vec{C}) = (\vec{A} \cdot \vec{C}) \vec{B} - (\vec{A} \cdot \vec{B}) \vec{C}$
as is clear from above, $\vec{A} \times (\vec{B} \times \vec{C})$ lies in plane of $\vec{B}$ and $\vec{C}$.

Useful to remember: $(\vec{A} \times \vec{B}) \cdot (\vec{C} \times \vec{D}) = (\vec{A} \cdot \vec{C})(\vec{B} \cdot \vec{D}) - (\vec{A} \cdot \vec{D})(\vec{B} \cdot \vec{C})$
 $(\vec{A} \times \vec{B}) \times (\vec{C} \times \vec{D}) = \left[\vec{A} \cdot (\vec{B} \times \vec{D}) \right] \vec{C} - \left[\vec{A} \cdot (\vec{B} \times \vec{C}) \right] \vec{D}$

Gradient Operator: $\nabla = \vec{\nabla} = \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \right)$

This **vector operator** is sometimes called the **nabla** or **del** operator.

Laplacian operator: $\nabla^2 = \nabla \cdot \nabla = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$

This is a **scalar operator**.

Differential: $f = f(x, y, z) \rightarrow df = \frac{\partial f}{\partial x} dx + \frac{\partial f}{\partial y} dy + \frac{\partial f}{\partial z} dz$

Chain Rule: If $x = x(t)$, $y = y(t)$ and $z = z(t)$ then $\frac{df}{dt} = \frac{\partial f}{\partial x} \frac{dx}{dt} + \frac{\partial f}{\partial y} \frac{dy}{dt} + \frac{\partial f}{\partial z} \frac{dz}{dt}$
If $x = x(s, t)$, $y = y(s, t)$ and $z = z(s, t)$ then $\frac{df}{ds} = \frac{\partial f}{\partial x} \frac{\partial x}{\partial s} + \frac{\partial f}{\partial y} \frac{\partial y}{\partial s} + \frac{\partial f}{\partial z} \frac{\partial z}{\partial s}$

Gradient Vector: $\nabla f = \text{grad} f = \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z} \right)$
the gradient vector at (x, y, z) is normal to the level surface through the point (x, y, z) .

Directional Derivative: The derivative of $f = f(x, y, z)$ in direction of $\vec{u}$ is
 $D_u f = \nabla f \cdot \frac{\vec{u}}{|\vec{u}|} = |\nabla f| \cos \theta$

Vector Field: $\vec{F}(\vec{x}) = (F_x, F_y, F_z) = F_x \hat{i} + F_y \hat{j} + F_z \hat{k}$
where $F_x = F_x(x, y, z)$, $F_y = F_y(x, y, z)$, and $F_z = F_z(x, y, z)$.

Divergence of Vector Field: $\text{div} \vec{F} = \nabla \cdot \vec{F} = \frac{\partial F_x}{\partial x} + \frac{\partial F_y}{\partial y} + \frac{\partial F_z}{\partial z}$
A vector field for which $\nabla \cdot \vec{F} = 0$ is called **solenoidal** or **divergence-free**.

Curl of Vector Field: $\text{curl} \vec{F} = \nabla \times \vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ F_x & F_y & F_z \end{vmatrix}$
A vector field for which $\nabla \times \vec{F} = 0$ is called **irrotational** or **curl-free**.

Laplacian of Vector Field: $\nabla^2 \vec{F} = (\nabla \cdot \nabla) \vec{F} = \nabla(\nabla \cdot \vec{F}) - \nabla \times (\nabla \times \vec{F})$
Note that $\nabla^2 \vec{F} \neq \nabla(\nabla \cdot \vec{F})$: **do not make this mistake**.

Let $S(\vec{x})$ and $T(\vec{x})$ be scalar fields, and let $\vec{A}(\vec{x})$ and $\vec{B}(\vec{x})$ be vector fields:

$\nabla S = \text{grad} S = \text{vector}$	$\nabla^2 S = \nabla \cdot (\nabla S) = \text{scalar}$
$\nabla \cdot \vec{A} = \text{div} \vec{A} = \text{scalar}$	$\nabla^2 \vec{A} = (\nabla \cdot \nabla) \vec{A} = \text{vector}$
$\nabla \times \vec{A} = \text{curl} \vec{A} = \text{vector}$	

$$\nabla \times (\nabla S) = 0 \quad \text{curl grad } S = 0$$

$$\nabla \cdot (\nabla \times \vec{A}) = 0 \quad \text{div curl } \vec{A} = 0$$

$$\nabla(ST) = S \nabla T + T \nabla S$$

$$\nabla \cdot (S\vec{A}) = S(\nabla \cdot \vec{A}) + \vec{A} \cdot \nabla S$$

$$\nabla \times (S\vec{A}) = (\nabla S) \times \vec{A} + S(\nabla \times \vec{A})$$

$$\nabla \cdot (\vec{A} \times \vec{B}) = \vec{B} \cdot (\nabla \times \vec{A}) - \vec{A} \cdot (\nabla \times \vec{B})$$

$$\nabla \times (\vec{A} \times \vec{B}) = \vec{A}(\nabla \cdot \vec{B}) - \vec{B}(\nabla \cdot \vec{A}) + (\vec{B} \cdot \nabla)\vec{A} - (\vec{A} \cdot \nabla)\vec{B}$$

$$\nabla(\vec{A} \cdot \vec{B}) = (\vec{A} \cdot \nabla)\vec{B} + (\vec{B} \cdot \nabla)\vec{A} + \vec{A} \times (\nabla \times \vec{B}) + \vec{B} \times (\nabla \times \vec{A})$$

$$\vec{A} \times (\nabla \times \vec{A}) = \frac{1}{2} \nabla(\vec{A} \cdot \vec{A}) - (\vec{A} \cdot \nabla)\vec{A}$$

$$\nabla \times (\nabla^2 \vec{A}) = \nabla^2(\nabla \times \vec{A})$$

Appendix B

Conservative Vector Fields

Line Integral of a Conservative Vector Field: Consider a curve γ running from location $\vec{x}_0$ to $\vec{x}_1$. Let $d\vec{l}$ be the directional element of length along γ (i.e., with direction equal to that of the tangent vector to γ), then, for any scalar field $\Phi(\vec{x})$,

$$\int_{\vec{x}_0}^{\vec{x}_1} \nabla \Phi \cdot d\vec{l} = \int_{\vec{x}_0}^{\vec{x}_1} d\Phi = \Phi(\vec{x}_1) - \Phi(\vec{x}_0)$$

This implies that the line integral is independent of γ , and hence

$$\oint_c \nabla \Phi \cdot d\vec{l} = 0$$

where c is a closed curve, and the integral is to be performed in the counter-clockwise direction.

Conservative Vector Fields:

A conservative vector field $\vec{F}$ has the following properties:

- $\vec{F}(\vec{x})$ is a gradient field, which means that there is a scalar field $\Phi(\vec{x})$ so that $\vec{F} = \nabla \Phi$
- Path independence: $\oint_c \vec{F} \cdot d\vec{l} = 0$
- Irrotational = curl-free: $\nabla \times \vec{F} = 0$

Appendix C

Integral Theorems

Green's Theorem: Consider a 2D vector field $\vec{F} = F_x \hat{i} + F_y \hat{j}$

$$\oint \vec{F} \cdot d\vec{l} = \int \int_A \nabla \times \vec{F} \cdot \hat{n} dA = \int \int_A |\nabla \times \vec{F}| dA$$

$$\oint \vec{F} \cdot \hat{n} dl = \int \int_A \nabla \cdot \vec{F} dA$$

NOTE: in the first equation we have used that $\nabla \times \vec{F}$ is always pointing in the direction of the normal $\hat{n}$.

Gauss' Divergence Theorem: Consider a 3D vector field $\vec{F} = (F_x, F_y, F_z)$

If S is a closed surface bounding a region D with normal pointing outwards, and $\vec{F}$ is a vector field defined and differentiable over all of D , then

$$\int \int_S \vec{F} \cdot d\vec{S} = \int \int \int_D \nabla \cdot \vec{F} dV$$

Stokes' Curl Theorem: Consider a 3D vector field $\vec{F} = (F_x, F_y, F_z)$

If C is a closed curve, and S is *any* surface bounded by C , then

$$\oint_c \vec{F} \cdot d\vec{l} = \int \int_S (\nabla \times \vec{F}) \cdot \hat{n} dS$$

NOTE: The curve of the line integral must have positive orientation, meaning that $d\vec{l}$ points counterclockwise when the normal of the surface points towards the viewer.

Appendix D

Curvi-Linear Coordinate Systems

In astrophysics, one often works in **curvi-linear**, rather than **Cartesian** coordinate systems. The two most often encountered examples are the **cylindrical** (R, ϕ, z) and **spherical** (r, θ, ϕ) coordinate systems.

In this appendix we describe how to handle **vector calculus** in non-Cartesian coordinate systems (Euclidean spaces only). After giving the ‘rules’ for arbitrary coordinate systems, we apply them to cylindrical and spherical coordinate systems, respectively.

Vector Calculus in an Arbitrary Coordinate System:

Consider a vector $\vec{x} = (x, y, z)$ in Cartesian coordinates. This means that we can write

$$\vec{x} = x \vec{e}_x + y \vec{e}_y + z \vec{e}_z$$

where $\vec{e}_x$, $\vec{e}_y$ and $\vec{e}_z$ are the unit directional vectors. Now consider the same vector $\vec{x}$, but expressed in another **general** (arbitrary) coordinate system; $\vec{x} = (q_1, q_2, q_3)$. It is tempting, **but terribly wrong**, to write that

$$\vec{x} = q_1 \vec{e}_1 + q_2 \vec{e}_2 + q_3 \vec{e}_3$$

where $\vec{e}_1$, $\vec{e}_2$ and $\vec{e}_3$ are the unit directional vectors in the new (q_1, q_2, q_3) -coordinate system. In what follows we show how to properly treat such generalized coordinate systems.

In general, one expresses the distance between (q_1, q_2, q_3) and $(q_1 + dq_1, q_2 + dq_2, q_3 + dq_3)$ in an arbitrary coordinate system as

$$ds = \sqrt{h_{ij} dq_i dq_j}$$

Here h_{ij} is called the **metric tensor**. In what follows, we will only consider **orthogonal** coordinate systems for which $h_{ij} = 0$ if $i \neq j$, so that $ds^2 = h_i^2 dq_i^2$ (Einstein summation convention) with $h_i = \sqrt{h_{ii}}$.

An example of an orthogonal coordinate system are the Cartesian coordinates, for which $h_{ij} = \delta_{ij}$. After all, the distance between two points separated by the infinitesimal displacement vector $d\vec{x} = (dx, dy, dz)$ is $ds^2 = |d\vec{x}|^2 = dx^2 + dy^2 + dz^2$.

The coordinates (x, y, z) and (q_1, q_2, q_3) are related to each other via the **transformation relations**

$$\begin{aligned}x &= x(q_1, q_2, q_3) \\y &= y(q_1, q_2, q_3) \\z &= z(q_1, q_2, q_3)\end{aligned}$$

and the corresponding inverse relations

$$\begin{aligned}q_1 &= q_1(x, y, z) \\q_2 &= q_2(x, y, z) \\q_3 &= q_3(x, y, z)\end{aligned}$$

Hence, we have that the **differential vector** is:

$$d\vec{x} = \frac{\partial \vec{x}}{\partial q_1} dq_1 + \frac{\partial \vec{x}}{\partial q_2} dq_2 + \frac{\partial \vec{x}}{\partial q_3} dq_3$$

where

$$\frac{\partial \vec{x}}{\partial q_i} = \frac{\partial}{\partial q_i}(x, y, z)$$

The **unit directional vectors** are:

$$\vec{e}_i = \frac{\partial \vec{x} / \partial q_i}{|\partial \vec{x} / \partial q_i|}$$

which allows us to rewrite the expression for the **differential vector** as

$$d\vec{x} = \left| \frac{\partial \vec{x}}{\partial q_1} \right| dq_1 \vec{e}_1 + \left| \frac{\partial \vec{x}}{\partial q_2} \right| dq_2 \vec{e}_2 + \left| \frac{\partial \vec{x}}{\partial q_3} \right| dq_3 \vec{e}_3$$

and thus

$$|d\vec{x}|^2 = \left| \frac{\partial \vec{x}}{\partial q_i} \right|^2 dq_i^2$$

(Einstein summation convention). Using the definition of the metric, according to which $|d\vec{x}|^2 = h_i^2 dq_i^2$ we thus infer that

$$\boxed{h_i = \left| \frac{\partial \vec{x}}{\partial q_i} \right|}$$

Using this expression for the metric allows us to write the **unit directional vectors** as

$$\vec{e}_i = \frac{1}{h_i} \frac{\partial \vec{x}}{\partial q_i}$$

and the **differential vector** in the compact form as

$$d\vec{x} = h_i dq_i \vec{e}_i$$

From the latter we also have that the **infinitesimal volume element** for a general coordinate system is given by

$$d^3\vec{x} = |h_1 h_2 h_3| dq_1 dq_2 dq_3$$

Note that the absolute values are needed to assure that $d^3\vec{x}$ is positive.

Now consider a vector $\vec{A}$. In the Cartesian **basis** $\mathcal{C} = \{\vec{e}_x, \vec{e}_y, \vec{e}_z\}$ we have that

$$[\vec{A}]_{\mathcal{C}} = A_x \vec{e}_x + A_y \vec{e}_y + A_z \vec{e}_z$$

In the **basis** $\mathcal{B} = \{\vec{e}_1, \vec{e}_2, \vec{e}_3\}$, corresponding to our generalized coordinate system, we instead have that

$$[\vec{A}]_{\mathcal{B}} = A_1 \vec{e}_1 + A_2 \vec{e}_2 + A_3 \vec{e}_3$$

We can rewrite the above as

$$[\vec{A}]_{\mathcal{B}} = A_1 \begin{pmatrix} e_{11} \\ e_{12} \\ e_{13} \end{pmatrix} + A_2 \begin{pmatrix} e_{21} \\ e_{22} \\ e_{23} \end{pmatrix} + A_3 \begin{pmatrix} e_{31} \\ e_{32} \\ e_{33} \end{pmatrix} = \begin{pmatrix} A_1 e_{11} + A_2 e_{21} + A_3 e_{31} \\ A_2 e_{12} + A_2 e_{22} + A_3 e_{32} \\ A_3 e_{13} + A_2 e_{23} + A_3 e_{33} \end{pmatrix}$$

and thus

$$[\vec{A}]_{\mathcal{B}} = \begin{pmatrix} e_{11} & e_{21} & e_{31} \\ e_{12} & e_{22} & e_{32} \\ e_{13} & e_{23} & e_{33} \end{pmatrix} \begin{pmatrix} A_1 \\ A_2 \\ A_3 \end{pmatrix} \equiv \mathbf{T} \begin{pmatrix} A_1 \\ A_2 \\ A_3 \end{pmatrix}$$

Using similar logic, one can write

$$[\vec{A}]_{\mathcal{C}} = \begin{pmatrix} e_{x1} & e_{y1} & e_{z1} \\ e_{x2} & e_{y2} & e_{z2} \\ e_{x3} & e_{y3} & e_{z3} \end{pmatrix} \begin{pmatrix} A_x \\ A_y \\ A_z \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} A_x \\ A_y \\ A_z \end{pmatrix} = \mathbf{I} \begin{pmatrix} A_x \\ A_y \\ A_z \end{pmatrix}$$

and since $\vec{A}$ is the same object independent of its basis we have that

$$\mathbf{I} \begin{pmatrix} A_x \\ A_y \\ A_z \end{pmatrix} = \mathbf{T} \begin{pmatrix} A_1 \\ A_2 \\ A_3 \end{pmatrix}$$

and thus, we see that the relation between $[\vec{A}]_{\mathcal{B}}$ and $[\vec{A}]_{\mathcal{C}}$ is given by

$$[\vec{A}]_{\mathcal{C}} = \mathbf{T} [\vec{A}]_{\mathcal{B}}, \quad [\vec{A}]_{\mathcal{B}} = \mathbf{T}^{-1} [\vec{A}]_{\mathcal{C}}$$

For this reason, $\mathbf{T}$ is called the **transformation of basis matrix**. Note that the columns of $\mathbf{T}$ are the unit-direction vectors $\vec{e}_i$, i.e., $T_{ij} = e_{ij}$. Since these are orthogonal to each other, the matrix $\mathbf{T}$ is said to be **orthogonal**, which implies that $\mathbf{T}^{-1} = \mathbf{T}^T$ (the inverse is equal to the transpose), and $\det(\mathbf{T}) = \pm 1$.

Now we are finally ready to determine how to write our position vector $\vec{x}$ in the new basis $\mathcal{B}$ of our generalized coordinate system. Let's write $\vec{x} = a_i \vec{e}_i$, i.e.

$$[\vec{x}]_{\mathcal{B}} = \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix}$$

We started this appendix by pointing out that it is tempting, but wrong, to set $a_i = q_i$ (as for the Cartesian basis). To see this, recall that $|\vec{x}| = \sqrt{(a_1)^2 + (a_2)^2 + (a_3)^2}$, from which it is immediately clear that each a_i needs to have the dimension of length. Hence, when q_i is an angle, clearly $a_i \neq q_i$. To compute the actual a_i you need to use the transformation of basis matrix as follows:

$$[\vec{x}]_{\mathcal{B}} = \mathbf{T}^{-1} [\vec{x}]_{\mathcal{C}} = \begin{pmatrix} e_{11} & e_{12} & e_{13} \\ e_{21} & e_{22} & e_{23} \\ e_{31} & e_{32} & e_{33} \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} e_{11}x + e_{12}y + e_{13}z \\ e_{21}x + e_{22}y + e_{23}z \\ e_{31}x + e_{32}y + e_{33}z \end{pmatrix}$$

Hence, using our expression for the unit direction vectors, we see that

$$\boxed{a_i = \frac{1}{h_i} \left(\frac{\partial x_j}{\partial q_i} x_j \right) = \frac{1}{h_i} \left(\frac{\partial \vec{x}}{\partial q_i} \cdot \vec{x} \right)}$$

Hence, the position vector in the generalized basis $\mathcal{B}$ is given by

$$[\vec{x}]_{\mathcal{B}} = \sum_i \frac{1}{h_i} \left(\frac{\partial \vec{x}}{\partial q_i} \cdot \vec{x} \right) \vec{e}_i$$

and by operating d/dt on $[\vec{x}]_{\mathcal{B}}$ we find that the corresponding velocity vector in the $\mathcal{B}$ basis is given by

$$[\vec{v}]_{\mathcal{B}} = \sum_i h_i \dot{q}_i \vec{e}_i$$

with $\dot{q}_i = dq_i/dt$. Note that the latter can also be inferred more directly by simply dividing the expression for the **differential vector** ($d\vec{x} = h_i dq_i \vec{e}_i$) by dt .

Next we write out the gradient, the divergence, the curl and the Laplacian for our generalized coordinate system:

The gradient:

$$\nabla\psi = \frac{1}{h_i} \frac{\partial\psi}{\partial q_i} \vec{e}_i$$

The divergence:

$$\nabla \cdot \vec{A} = \frac{1}{h_1 h_2 h_3} \left[\frac{\partial}{\partial q_1} (h_2 h_3 A_1) + \frac{\partial}{\partial q_2} (h_3 h_1 A_2) + \frac{\partial}{\partial q_3} (h_1 h_2 A_3) \right]$$

The curl (only one component shown):

$$(\nabla \times \vec{A})_3 = \frac{1}{h_1 h_2} \left[\frac{\partial}{\partial q_1} (h_2 A_2) - \frac{\partial}{\partial q_2} (h_1 A_1) \right]$$

The Laplacian:

$$\nabla^2 \psi = \frac{1}{h_1 h_2 h_3} \left[\frac{\partial}{\partial q_1} \left(\frac{h_2 h_3}{h_1} \frac{\partial \psi}{\partial q_1} \right) + \frac{\partial}{\partial q_2} \left(\frac{h_3 h_1}{h_2} \frac{\partial \psi}{\partial q_2} \right) + \frac{\partial}{\partial q_3} \left(\frac{h_1 h_2}{h_3} \frac{\partial \psi}{\partial q_3} \right) \right]$$

The Convective operator:

$$(\vec{A} \cdot \nabla) \vec{B} = \left[\frac{A_i}{h_i} \frac{\partial B_j}{\partial q_i} + \frac{B_i}{h_i h_j} \left(A_j \frac{\partial h_j}{\partial q_i} - A_i \frac{\partial h_i}{\partial q_j} \right) \right] \vec{e}_j$$

Vector Calculus in Cylindrical Coordinates:

For cylindrical coordinates (R, ϕ, z) we have that

$$x = R \cos \phi \quad y = R \sin \phi \quad z = z$$

The **scale factors** of the **metric** therefore are:

$$h_R = 1 \quad h_\phi = R \quad h_z = 1$$

and the position vector is $\vec{x} = R\vec{e}_R + z\vec{e}_z$.

Let $\vec{A} = A_R\vec{e}_R + A_\phi\vec{e}_\phi + A_z\vec{e}_z$ an arbitrary vector, then

$$\begin{aligned} A_R &= A_x \cos \phi - A_y \sin \phi \\ A_\phi &= -A_x \sin \phi + A_y \cos \phi \\ A_z &= A_z \end{aligned}$$

In cylindrical coordinates the **velocity vector** becomes:

$$\begin{aligned} \vec{v} &= \dot{R}\vec{e}_R + R\dot{\vec{e}}_R + \dot{z}\vec{e}_z \\ &= \dot{R}\vec{e}_R + R\dot{\phi}\vec{e}_\phi + \dot{z}\vec{e}_z \end{aligned}$$

The Gradient:

$$\nabla \cdot \vec{A} = \frac{1}{R} \frac{\partial}{\partial R}(RA_R) + \frac{1}{R} \frac{\partial A_\phi}{\partial \phi} + \frac{\partial A_z}{\partial z}$$

The Convective Operator:

$$\begin{aligned} (\vec{A} \cdot \nabla) \vec{B} &= \left(A_R \frac{\partial B_R}{\partial R} + \frac{A_\phi}{R} \frac{\partial B_R}{\partial \phi} + A_z \frac{\partial B_R}{\partial z} - \frac{A_\phi B_\phi}{R} \right) \vec{e}_R \\ &+ \left(A_R \frac{\partial B_\phi}{\partial R} + \frac{A_\phi}{R} \frac{\partial B_\phi}{\partial \phi} + A_z \frac{\partial B_\phi}{\partial z} + \frac{A_\phi B_R}{R} \right) \vec{e}_\phi \\ &+ \left(A_R \frac{\partial B_z}{\partial R} + \frac{A_\phi}{R} \frac{\partial B_z}{\partial \phi} + A_z \frac{\partial B_z}{\partial z} \right) \vec{e}_z \end{aligned}$$

The Laplacian:

$$\text{scalar :} \quad \nabla^2 \psi = \frac{1}{R} \frac{\partial}{\partial R} \left(R \frac{\partial \psi}{\partial R} \right) + \frac{1}{R^2} \frac{\partial^2 \psi}{\partial \phi^2} + \frac{\partial^2 \psi}{\partial z^2}$$

$$\begin{aligned} \text{vector :} \quad \nabla^2 \vec{F} &= \left(\nabla^2 F_R - \frac{F_R}{R^2} - \frac{2}{R^2} \frac{\partial F_\theta}{\partial \theta} \right) \vec{e}_R \\ &+ \left(\nabla^2 F_\theta + \frac{2}{R^2} \frac{\partial F_R}{\partial \theta} - \frac{F_\theta}{R^2} \right) \vec{e}_\theta \\ &+ (\nabla^2 F_z) \vec{e}_z \end{aligned}$$

Vector Calculus in Spherical Coordinates:

For spherical coordinates (r, θ, ϕ) we have that

$$x = r \sin \theta \cos \phi \quad y = r \sin \theta \sin \phi \quad z = r \cos \theta$$

The **scale factors** of the **metric** therefore are:

$$h_r = 1 \quad h_\theta = r \quad h_\phi = r \sin \theta$$

and the position vector is $\vec{x} = r\vec{e}_r$.

Let $\vec{A} = A_r\vec{e}_r + A_\theta\vec{e}_\theta + A_\phi\vec{e}_\phi$ an arbitrary vector, then

$$\begin{aligned} A_r &= A_x \sin \theta \cos \phi + A_y \sin \theta \sin \phi + A_z \cos \theta \\ A_\theta &= A_x \cos \theta \cos \phi + A_y \cos \theta \sin \phi - A_z \sin \theta \\ A_\phi &= -A_x \sin \phi + A_y \cos \phi \end{aligned}$$

In spherical coordinates the **velocity vector** becomes:

$$\begin{aligned} \vec{v} &= \dot{r} \vec{e}_r + r \dot{\vec{e}}_r \\ &= \dot{r} \vec{e}_r + r \dot{\theta} \vec{e}_\theta + r \sin \theta \dot{\phi} \vec{e}_\phi \end{aligned}$$

The Gradient:

$$\nabla \cdot \vec{A} = \frac{1}{r^2} \frac{\partial}{\partial r} (r^2 A_r) + \frac{1}{r \sin \theta} \frac{\partial}{\partial \theta} (\sin \theta A_\theta) + \frac{1}{r \sin \theta} \frac{\partial A_\phi}{\partial \phi}$$

The Convective Operator:

$$\begin{aligned} (\vec{A} \cdot \nabla) \vec{B} &= \left(A_r \frac{\partial B_r}{\partial r} + \frac{A_\theta}{r} \frac{\partial B_r}{\partial \theta} + \frac{A_\phi}{r \sin \theta} \frac{\partial B_r}{\partial \phi} - \frac{A_\theta B_\theta + A_\phi B_\phi}{r} \right) \vec{e}_r \\ &+ \left(A_r \frac{\partial B_\theta}{\partial r} + \frac{A_\theta}{r} \frac{\partial B_\theta}{\partial \theta} + \frac{A_\phi}{r \sin \theta} \frac{\partial B_\theta}{\partial \phi} + \frac{A_\theta B_r}{r} - \frac{A_\phi B_\phi \cot \theta}{r} \right) \vec{e}_\theta \\ &+ \left(A_r \frac{\partial B_\phi}{\partial r} + \frac{A_\theta}{r} \frac{\partial B_\phi}{\partial \theta} + \frac{A_\phi}{r \sin \theta} \frac{\partial B_\phi}{\partial \phi} + \frac{A_\phi B_r}{r} + \frac{A_\phi B_\theta \cot \theta}{r} \right) \vec{e}_\phi \end{aligned}$$

The Laplacian:

$$\text{scalar :} \quad \nabla^2 \psi = \frac{1}{r^2} \frac{\partial}{\partial r} \left(r^2 \frac{\partial \psi}{\partial r} \right) + \frac{1}{r^2 \sin \theta} \frac{\partial}{\partial \theta} \left(\sin \theta \frac{\partial \psi}{\partial \theta} \right) + \frac{1}{r^2 \sin^2 \theta} \frac{\partial^2 \psi}{\partial \phi^2}$$

$$\begin{aligned} \text{vector :} \quad \nabla^2 \vec{F} &= \left(\nabla^2 F_r - \frac{2F_r}{r^2} - \frac{2}{r^2 \sin \theta} \frac{\partial(F_\theta \sin \theta)}{\partial \theta} - \frac{2}{r^2 \sin \theta} \frac{\partial F_\phi}{\partial \phi} \right) \vec{e}_r \\ &+ \left(\nabla^2 F_\theta + \frac{2}{r^2} \frac{\partial F_r}{\partial \theta} - \frac{F_\theta}{r^2 \sin \theta} - \frac{2 \cos \theta}{r^2 \sin^2 \theta} \frac{\partial F_\phi}{\partial \phi} \right) \vec{e}_\theta \\ &+ \left(\nabla^2 F_\phi + \frac{2}{r^2 \sin \theta} \frac{\partial F_r}{\partial \phi} + \frac{2 \cos \theta}{r^2 \sin^2 \theta} \frac{\partial F_\theta}{\partial \phi} - \frac{F_\phi}{r^2 \sin^2 \theta} \right) \vec{e}_\phi \end{aligned}$$

Appendix E

The Levi-Civita Symbol

The **Levi-Civita symbol**, also known as the **permutation symbol** or the **anti-symmetric symbol**, is a collection of numbers, defined from the sign of a permutation of the natural numbers $1, 2, 3, \dots, n$. It is often encountered in linear algebra, vector and tensor calculus, and differential geometry.

The n -dimensional Levi-Civita symbol is indicated by $\varepsilon_{i_1 i_2 \dots i_n}$, where each index $i_1, i_2, \dots, i_n$ takes values $1, 2, \dots, n$, and has the defining property that the symbol is total antisymmetric in all its indices: when any two indices are interchanged, the symbol is negated:

$$\varepsilon_{\dots i_p \dots i_q \dots} = -\varepsilon_{\dots i_q \dots i_p \dots}$$

If any two indices are equal, the symbol is zero, and when all indices are unequal, we have that

$$\varepsilon_{i_1 i_2 \dots i_n} = (-1)^p \varepsilon_{1, 2, \dots, n}$$

where p is called the **parity of the permutation**. It is the number of pairwise interchanges necessary to unscramble $i_1, i_2, \dots, i_n$ into the order $1, 2, \dots, n$. A permutation is said to be even (odd) if its parity is an even (odd) number.

Example: what is the parity of $\{3, 4, 5, 2, 1\}$?

$$\begin{aligned} &\{1, 2, 3, 4, 5\} \\ &\{\mathbf{3}, 2, \mathbf{1}, 4, 5\} \\ &\{3, \mathbf{4}, 1, \mathbf{2}, 5\} \\ &\{3, 4, \mathbf{5}, 2, \mathbf{1}\} \end{aligned}$$

Answer: $p = 3$, since three pairwise interchanges are required.

In three dimensions the Levi-Civita symbol is defined by

$$\varepsilon_{ijk} = \begin{cases} +1 & \text{if } (i, j, k) \text{ is } (1, 2, 3), (2, 3, 1), \text{ or } (3, 1, 2) \\ -1 & \text{if } (i, j, k) \text{ is } (3, 2, 1), (1, 3, 2), \text{ or } (2, 1, 3) \\ 0 & \text{if } i = j, \text{ or } j = k, \text{ or } k = i \end{cases}$$

Appendix F

The Viscous Stress Tensor

As discussed in Chapter 4, the deviatoric stress tensor, τ_{ij} , is only non-zero in the presence of shear in the fluid flow. This suggests that

$$\tau_{ij} = T_{ijkl} \frac{\partial u_k}{\partial x_l}$$

where T_{ijkl} is a proportionality tensor of **rank** four. In what follows we derive an expression for T_{ijkl} . We start by noting that since σ_{ij} is symmetric, we also have that τ_{ij} will be symmetric. Hence, we expect that the above dependence can only involve the **symmetric** component of the deformation tensor, $\mathcal{T}_{kl} = \partial u_k / \partial x_l$. Hence, it is useful to split the deformation tensor in its **symmetric** and **anti-symmetric** components:

$$\frac{\partial u_i}{\partial x_j} = e_{ij} + \xi_{ij}$$

where

$$\begin{aligned} e_{ij} &= \frac{1}{2} \left[\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} \right] \\ \xi_{ij} &= \frac{1}{2} \left[\frac{\partial u_i}{\partial x_j} - \frac{\partial u_j}{\partial x_i} \right] \end{aligned}$$

The symmetric part of the deformation tensor, e_{ij} , is called the **rate of strain tensor**, while the anti-symmetric part, ξ_{ij} , expresses the **vorticity** $\vec{w} \equiv \nabla \times \vec{u}$ in the velocity field, i.e., $\xi_{ij} = -\frac{1}{2} \varepsilon_{ijk} w_k$. Note that one can always find a coordinate system for which e_{ij} is diagonal. The axes of that coordinate frame indicate the **eigendirections** of the strain (compression or stretching) on the fluid element.

In terms of the relation between the viscous stress tensor, τ_{ij} , and the deformation tensor, $\mathcal{T}_{kl}$, there are a number of properties that are important.

- **Locality:** the $\tau_{ij} - \mathcal{T}_{kl}$ -relation is said to be **local** if the stress tensor is only a function of the deformation tensor and thermodynamic state functions like temperature.
- **Homogeneity:** the $\tau_{ij} - \mathcal{T}_{kl}$ -relation is said to be **homogeneous** if it is everywhere the same. The viscous stress tensor may depend on location $\vec{x}$ only insofar as $\mathcal{T}_{ij}$ or the thermodynamic state functions depend on $\vec{x}$. This distinguishes a fluid from a solid, in which the stress tensor depends on the stress itself.
- **Isotropy:** the $\tau_{ij} - \mathcal{T}_{kl}$ -relation is said to be **isotropic** if it has no preferred direction.
- **Linearity:** the $\tau_{ij} - \mathcal{T}_{kl}$ -relation is said to be **linear** if the relation between the stress and rate-of-strain is linear. This is equivalent to saying that τ_{ij} does not depend on $\nabla^2 \vec{u}$ or higher-order derivatives.

A fluid that is local, homogeneous and isotropic is called a **Stokesian** fluid. A Stokesian fluid that is linear is called a **Newtonian** fluid. Experiments have shown that most (astrophysical) fluids are Newtonian to good approximation. Hence, in what follows we will assume that our fluids are Newtonian, unless specifically stated otherwise. For a Newtonian fluid, it can be shown (using linear algebra) that the most general form of our proportionality tensor is given by

$$T_{ijkl} = \lambda \delta_{ij} \delta_{kl} + \mu (\delta_{ik} \delta_{jl} + \delta_{il} \delta_{jk})$$

Hence, for a **Newtonian** fluid the **viscous stress tensor** is

$$\tau_{ij} = 2\mu e_{ij} + \lambda e_{kk} \delta_{ij}$$

where μ is the **coefficient of shear viscosity**, λ is a scalar, δ_{ij} is the Kronecker delta function, and $e_{kk} = \text{Tr}(e) = \partial u_k / \partial x_k = \nabla \cdot \vec{u}$ (summation convention).

Note that (in a Newtonian fluid) the viscous stress tensor depends **only** on the symmetric component of the deformation tensor (the rate-of-strain tensor e_{ij}), but **not** on the antisymmetric component which describes **vorticity**. You can understand the fact that viscosity and vorticity are unrelated by considering a fluid disk in solid body rotation (i.e., $\nabla \cdot \vec{u} = 0$ and $\nabla \times \vec{u} = \vec{\omega} \neq 0$). In such a fluid there is no "slippage", hence no shear, and therefore no manifestation of viscosity.

Thus far we have derived that the stress tensor, σ_{ij} , which in principle has 6 unknowns, can be reduced to a function of three unknowns only (P , μ , λ) as long as the fluid is **Newtonian**. Note that these three scalars, in general, are functions of temperature and density. We now focus on these three scalars in more detail, starting with the pressure P . To be exact, P is the **thermodynamic equilibrium pressure**, and is normally computed thermodynamically from some equation of state, $P = P(\rho, T)$. It is related to the translational kinetic energy of the particles when the fluid, in equilibrium, has reached **equipartition** of energy among **all** its degrees of freedom, including (in the case of molecules) rotational and vibrations degrees of freedom.

In addition to the thermodynamic equilibrium pressure, P , we can also define a **mechanical pressure**, P_m , which is purely related to the translational motion of the particles, **independent** of whether the system has reached full equipartition of energy. The mechanical pressure is simply the average normal stress and therefore follows from the stress tensor according to

$$P_m = -\frac{1}{3} \text{Tr}(\sigma_{ij}) = -\frac{1}{3} (\sigma_{11} + \sigma_{22} + \sigma_{33})$$

Using that

$$\sigma_{ij} = -P \delta_{ij} + 2\mu e_{ij} + \lambda e_{kk} \delta_{ij}$$

we thus obtain the following relation between the two pressures:

$$P_m = P - \eta \nabla \cdot \vec{u}$$

where

$$\eta = \frac{2}{3}\mu + \lambda = \frac{P - P_m}{\nabla \cdot \vec{u}}$$

is the **coefficient of bulk viscosity**. We can now write the stress tensor as

$$\sigma_{ij} = -P \delta_{ij} + \mu \left[\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} - \frac{2}{3} \delta_{ij} \frac{\partial u_k}{\partial x_k} \right] + \eta \delta_{ij} \frac{\partial u_k}{\partial x_k}$$

This is the full expression for the stress tensor in terms of the coefficients of **shear viscosity**, μ , and **bulk viscosity**, η .

Appendix G

The Chemical Potential

Consider a system which can exchange energy **and particles** with a reservoir, and the volume of which can change. There are three ways for this system to increase its internal energy; heating, changing the system's volume (i.e., doing work on the system), or adding particles. Hence,

$$dU = T dS - P dV + \mu dN$$

Note that this is the first law of thermodynamics, but now with the added possibility of changing the number of particles of the system. The scalar quantity μ is called the **chemical potential**, and is defined by

$$\mu = \left(\frac{\partial U}{\partial N} \right)_{S,V}$$

This is not to be confused with the μ used to denote the mean weight per particle, which ALWAYS appears in combination with the proton mass, m_p . As is evident from the above expression, the chemical potential quantifies how the internal energy of the system changes if particles are added or removed, while keeping the entropy and volume of the system fixed. The chemical potential appears in the Fermi-Dirac distribution describing the momentum distribution of a gas of fermions or bosons.

Consider an ideal gas, of volume V , entropy S and with internal energy U . Now imagine adding a particle of zero energy ($\epsilon = 0$), while keeping the volume fixed. Since $\epsilon = 0$, we also have that $dU = 0$. But what about the entropy? Well, we have increased the number of ways in which we can redistribute the energy U (a macrostate quantity) over the different particles (different microstates). Hence, by adding this particle we have increased the system's entropy. If we want to add a particle while keeping S fixed, we need to decrease U to offset the increase in the number of 'degrees of freedom' over which to distribute this energy. Hence, keeping S (and V) fixed, requires that the particle has negative energy, and we thus see that $\mu < 0$.

For a fully degenerate Fermi gas, we have that $T = 0$, and thus $S = 0$ (i.e., there is only one micro-state associated with this macrostate, and that is the fully degenerate one). If we now add a particle, and demand that we keep $S = 0$, then that particle must have the Fermi energy (see Chapter 6); $\epsilon = E_f$. Hence, for a fully degenerate gas, $\mu = E_f$.

Finally, consider a photon gas in thermal equilibrium inside a container. Contrary to an ideal gas, in a photon gas the number of particles (photons) cannot be arbitrary. The number of photons at given temperature, T , and thus at given U , is given by the Planck distribution and is constantly adjusted (through absorption and emission against the wall of the container) so that the photon gas remains in thermal equilibrium. In other words, N_γ is not a degree of freedom for the system, but it is set by the volume and the temperature of the gas. Since we can't change N while maintaining S (or T) and V , we have that $\mu = 0$ for photons.

To end this discussion of the chemical potential, we address the origin of its name, which may, at first, seem weird. Let's start with the 'potential' part. The origin of this name is clear from the following. According to its definition (see above), the chemical potential is the 'internal energy' per unit amount (moles). Now consider the following correspondences:

Gravitational potential is the gravitational energy per unit mass:

$$W = \frac{G m_1 m_2}{r} \quad \Rightarrow \quad \phi = \frac{G m}{r} \quad \Rightarrow \quad \phi = \frac{\partial W}{\partial m}$$

Similarly, electrical potential is the electrical energy per unit charge

$$V = \frac{1}{4\pi\epsilon_0} \frac{q_1 q_2}{r} \quad \Rightarrow \quad \phi = \frac{1}{4\pi\epsilon_0} \frac{q}{r} \quad \Rightarrow \quad \phi = \frac{\partial V}{\partial q}$$

These examples make it clear why μ is considered a 'potential'. Finally, the word chemical arises from the fact that the μ plays an important role in chemistry (i.e., when considering systems in which chemical reactions take place, which change the particles). In this respect, it is important to be aware of the fact that μ is an additive quantity that is conserved in a chemical reaction. Hence, for a chemical

reaction $i + j \rightarrow k + l$ one has that $\mu_i + \mu_j = \mu_k + \mu_l$. As an example, consider the annihilation of an electron and a positron into two photons. Using that $\mu = 0$ for photons, we see that the chemical potential of elementary particles (i.e., electrons) must be opposite to that of their anti-particles (i.e., positrons).

Because of the additive nature of the chemical potential, we also have that the above equation for dU changes slightly whenever the gas consists of different particle species; it becomes

$$dU = T dS - P dV + \sum_i \mu_i dN_i$$

where the summation is over all species i . If the gas consists of equal numbers of elementary particles and anti-particles, then the total chemical potential of the system will be equal to zero. In fact, in many treatments of fluid dynamics it may be assumed that $\sum_i \mu_i dN_i = 0$; in particular when the relevant reactions are ‘frozen’ (i.e., occur on a timescales τ_{react} that are much longer than the dynamical timescales τ_{dyn} of interest), so that $dN_i = 0$, or if the reactions go so fast ($\tau_{\text{react}} \ll \tau_{\text{dyn}}$) that each reaction and its inverse are in local thermodynamic equilibrium, in which case $\sum_i \mu_i dN_i = 0$ for those species involved in the reaction. Only in the rare, intermediate case when $\tau_{\text{react}} \sim \tau_{\text{dyn}}$ is it important to keep track of the relative abundances of the various chemical and/or nuclear species.